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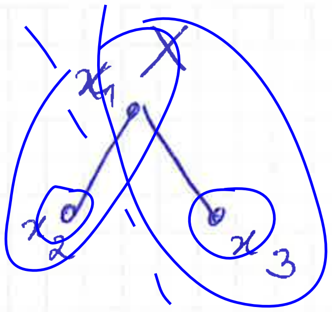
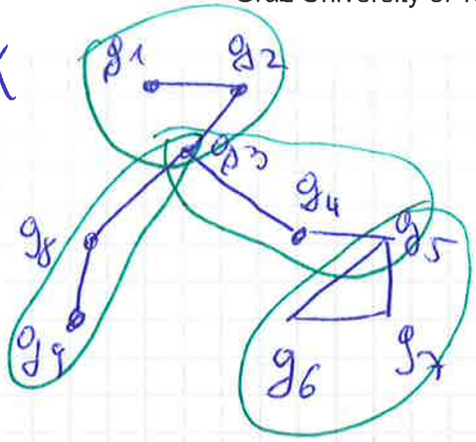
$X \equiv H$



$G = IX$



$V_{t_1} = \{g_5, g_6, g_7\}$

$f(g_i) = x_3 \quad \forall i \in \overline{4,7}$

$f(g_i) = x_1 \quad \forall i \in \overline{1,3}$

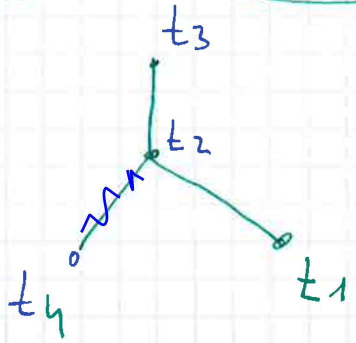
$f(g_i) = x_2 \quad \forall i \in \overline{8,9}$

$U_{x_3} = \{g_4, g_5, g_6, g_7\}$

$U_{x_1} = \{g_1, g_2, g_3\}$

$U_{x_2} = \{g_8, g_9\}$

T



$V_{t_1} = \{g_5, g_6, g_7\}$

$V_{t_2} = \{g_3, g_4, g_5\}$

$V_{t_3} = \{g_1, g_2, g_3\}$

$V_{t_4} = \{g_3, g_8, g_9\}$

$(T, (V_{t_i})_{i \in \overline{1,4}})$ is a tree decomposition of G

Construct a tree decomposition $(T, W = (W_t)_{t \in \overline{1,4}})$ of X.

$W_{t_1} = \{f(g) : g \in V_{t_1}\} = \{f(g_5), f(g_6), f(g_7)\} = \{x_3\}$

$W_{t_2} = \{f(g) : g \in V_{t_2}\} = \{f(g_3), f(g_4), f(g_5)\} = \{x_3, x_1\}$

$W_{t_3} = \{f(g) : g \in V_{t_3}\} = \{f(g_1), f(g_2), f(g_3)\} = \{x_1\}$

$W_{t_4} = \{f(g) : g \in V_{t_4}\} = \{f(g_3), f(g_8), f(g_9)\} = \{x_2, x_1\}$

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Consider now another useful consequence of Lemma 8.9: (resp. Lemma)

Lemma 8.12 Given a set $W \subseteq V(G)$, there is either (a) a $t \in T$ such $W \subseteq V_t$ or (b) there are vertices $w_1, w_2 \in W$ and an edge $\{t_1 t_2\} \in E(T)$ such that w_1 or w_2 lie outside the set $V_{t_1} \cap V_{t_2}$ and are separated from it in G. (Here $(T, (V_t)_{t \in T})$ is a tree decomposition of G.

Proof ⇒ We orient the edges of T in a particular way

$\forall \{t_1 t_2\} \in E(T)$ let T_1, T_2 be the subtrees in the forest $T \setminus \{t_1 t_2\}$ such that $t_1 \in V(T_1)$, $t_2 \in V(T_2)$ and let $U_i = \bigcup_{t \in V(T_i)} V_t$, $i \in \overline{1,2}$ (like in Lemma 8.9)

Then by Lemma 8.9 $V_{t_1} \cap V_{t_2}$ separates U_1 from U_2

⇒ Assume now that (b) in the statement of the Lemma is not fulfilled, i.e

$\forall w_1, w_2 \in W$ $\forall \{t_1, t_2\} \in E(T)$ ~~such~~ that $w_1, w_2 \notin V_{t_1} \cap V_{t_2}$ and w_1, w_2 are not separated from $V_{t_1} \cap V_{t_2}$ in G.

Both then $w_1, w_2 \in U_1$ or $w_1, w_2 \in U_2$ (by Lemma 8.9)

Thus $\exists i \in \{1, 2\}$ such that $w_1, w_2 \in U_i$. $w_1, w_2 \in U_1$

Then $\{t_1, t_2\}$ is oriented towards t_i. $t_1 \leftarrow t_2$

→ Let t the last vertex of a maximal directed path in T. We claim that $W \subseteq V_t$, thus (a) holds.

directed path t

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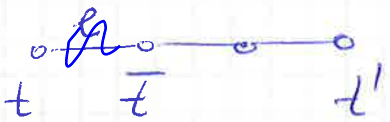
Let $w \in W$ and let $t' \in T$ be such that $w \in V_{t'}$. If $t' \neq t$, consider the edge e with $t \in e$ which separates t from t' in T



$\bar{t} = t'$ could happen!

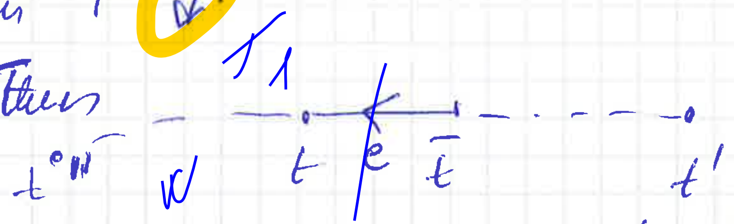
This edge e is directed from $\bar{t}$ to t $\quad t \leftarrow \bar{t}$

Indeed, there is an edge oriented as $\longrightarrow t$

If $\bar{t} \rightarrow t$ is this edge we are done.

Otherwise the edge e looks as $t \xrightarrow[e]{} \bar{t}$ and there is an edge $\bar{t}' \rightarrow t$ also, so t is not the vertex of a maximal directed path in T $\lightning$.

Thus



Since we have assumed that (b) is violated $V_t \cap V_{\bar{t}}$ does not separate any vertices $w_1, w_2 \in W$. $\Rightarrow$ All vertices of W "lie on one side of $T \setminus \{t, \bar{t}\}$, and by construction on that side toward which $\{t, \bar{t}\}$ is directed.

So $w \in U_1 = \bigcup_{x \in V(T_1)} V_x$ holds where T_1 is the subtree of $T \setminus \{t, \bar{t}\}$ containing t.

So $w \in V_{t''}$ for some $t'' \in V(T_1)$

By (T_3) $w \in V_t$ because t is the t''-t'-path in T.

This holds $\forall\, w \in W \implies W \subseteq V_t$.

Corollary 8.13 Any complete subgraph of G is contained in some part of a tree decomposition $(T, \mathcal{V})$ of G.

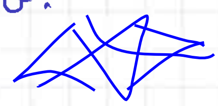
Observe

As we may have observed in all picture of tree decompositions we have seen so far the parts of $(T, \mathcal{V})$ reflect the structure of the tree T, so G resembles T in this sense. But this however valuable only to the extent that the structure of G within each part is negligible, i.e the smaller the parts the closer the resemblance.

Def: Let G be a graph with a tree decomposition $(T, \mathcal{V})$. The width of $(T, \mathcal{V})$ is given as

$$\max\{|V_t| - 1 : t \in T\}$$

The tree-width $tw(G)$ of G is the smallest width of any tree-decomposition of G.

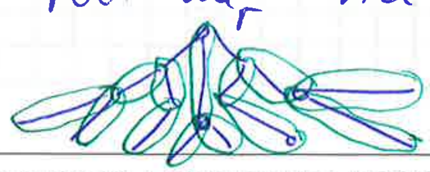


Remark 1) $tw(G)$ is well defined.

Indeed every graph G has a trivial tree decomposition: set $V(T) = \{x\}$, $E(T) = \emptyset$, hence T is a singleton, and $V_x = V(G)$, thus $\mathcal{V} = \{V_x = V(G)\}$. The width of this decomposition is $|V(G)| - 1$.

2) A tree T has $tw(T) = 1$, unless $|T| = 1$! $(T, \mathcal{V})$

Indeed root the tree at some vertex r and contract the branch sets iteratively starting with the leaves.



For every vertex $t \in V(T) \setminus \{r\}$, $V_t = \{t, p(t)\}$, where p is the parent of T; $V_r = \{r\}$.

$\mathcal{V}_t = V_t \cap \ldots$

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3) By lemma 8.10 and lemma 8.11 the tree-width of a graph will never be increased by deletion or contraction.

Proposition 8.14 If $H \preceq G$ then $tw(H) \leq tw(G)$, where $\preceq$ is the minor relation.

Theorem 8.15 (Robertson & Seymour 1990)
$\forall$ integer $k > 0$ the graphs of tree width $< k$ are well-quasi-ordered by the minor relation.
Without a proof (a proof can be read in chapter 12 of R. Diestel, Graph theory, Springer, 2012.

Proposition 8.16 G is chordal iff G has a tree decomposition into complete parts.

Proof Induction on $|G|$.
$\Rightarrow$ Assume that G has a tree-decomposition $(T, \mathcal{V})$ such that $G[V_t]$ is complete $\forall t \in T$. Choose such a $(T, \mathcal{V})$ with $|T|$ minimal. If $|T| = 1$ then $V_t = V(G)$ and hence G is complete $\Rightarrow$ chordal. So assume w.l.o.g. $|T| \geq 2$. Let $\{t_1, t_2\} \in E(T)$; let T_1, T_2 be the trees of the forest $T \setminus \{t_1 t_2\}$ such that $t_i \in V(T_i)$, $i \in \overline{1,2}$, and let $U_i = \bigcup_{t \in V(T_i)} V_t$, $i \in \overline{1,2}$. Let $G_i = G[U_i]$, $i \in \overline{1,2}$.

Then $G = G_1 \cup G_2$, i.e. $V(G) = V(G_1) \cup V(G_2)$
$E(G) = E(G_1) \cup E(G_2)$

by (T1), (T2).
Moreover, $V(G_1 \cap G_2) = V_{t_1} \cap V_{t_2}$ by lemma 8.5
(separation lemma)

$\Rightarrow G_1 \cap G_2$ is complete.

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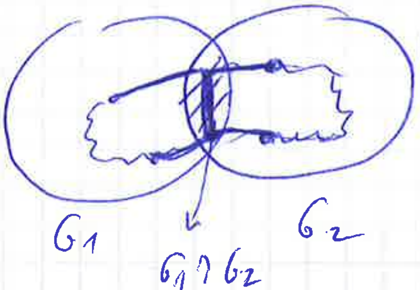
Since $(T_i, \mathcal{V}_i = (V_t)_{t \in T_i})$ is a tree decomposition of G_i in complete parts, $(i \in \overline{1,2})$, by the induction hypothesis both G_i, $i \in \overline{1,2}$, are chordal. So any cycle in G_i, $i \in \overline{1,2}$, has a chord.



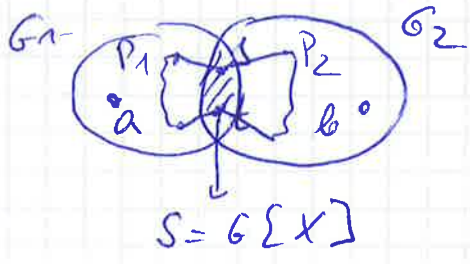
Any cycle C in G with vertices in G_1 and G_2 has to pass through $G_1 \cap G_2$ and since $G_1 \cap G_2$ is complete the cycle C must have a chord.

$\Rightarrow$ G is chordal.

$\Leftarrow$ Assume that G is chordal. If G is complete, there is nothing to show. So assume that G is not complete. Let $a, b \in V(G)$ with $\{a,b\} \notin E(G)$.

Claim: G is the union of smaller chordal graphs G_1, G_2 with $S := G_1 \cap G_2$ complete.

Let $X \subseteq V(G) \setminus \{a,b\}$ be a minimal a-b-separator. Let C be the component of $G - X$ containing a. and let $G_1 := G[V(C) \cup X]$ and $G_2 := G[V(G) \setminus C]$



G_i, $i \in \overline{1,2}$, are chordal as subgraphs of G (chordal)

So we show $S = G[X]$ is complete. Let $s, t \in X$ and assume $\{s,t\} \notin E(G)$

Since X is a minimal a-b-separator both s and t have neighbors in C and also in $G - G_1$.

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Let P_i be a shortest path from $?$ to t in G_i, $i \in \overline{1,2}$, with all inner vertices in $G_i \setminus X$.
$\Rightarrow P_1 \cup P_2$ is a chordless cycle of length ≥ 4 in G ↯

end proof of the claim ☐

Now apply the induction hypothesis on G_i, $i \in \overline{1,2}$.
Let (T_i, V_i) be a tree decomp. of G_i with complete parts, $\forall i \in \overline{1,2}$. By ~~theorem~~ Corollary 8.13 the complete subgraph $S = G_1 \cap G_2$ lies inside one of those parts say in V_{t_i} with $t_i \in V(T_i)$, $i \in \overline{1,2}$.
Then it is easy to check that $(T_1 \cup T_2 \cup \{t_1 t_2\}, V_1 \cup V_2)$ is a tree-decomposition of G into complete parts.
(homework!)

☐

Corollary 8.17 For any graph G

$$tw(G) = \min \{ \omega(H) - 1 : G \subseteq H, \; G \text{ is chordal} \}$$

Proof $\leq$ By Prop. 8.16 and Corollary 8.13 $\forall H$ chordal there exists a tree-decompos. of width $\omega(H) - 1$. Indeed. Prop. 8.16 $\Rightarrow$ $\exists (T, V)$ with complete parts $\Rightarrow$ every part is a clique in H $\Rightarrow$ width$(T, V) \leq \omega(H) - 1$.
On the other hand by Corollary 8.13 the max clique of G belongs to a part and indeed coincides with a part thus $\omega(H) - 1 \leq$ width(T,V)
So width$(T, V) = \omega(H) - 1$.
Every such tree-decomp. of some H induces a tree-decomp. of G, i.e. $tw(G) \leq \omega(H) - 1$ $\forall H$ chordal with $G \subseteq H$

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$\geq$ On the other hand, let us construct an H chordal with $G \subseteq H$ and $tw(G) \geq \omega(H) - 1$.

Let $(T, \mathcal{V})$ be a tree-decomp. of G of width $tw(G)$.

$\forall\, t \in T$ let K_t be the complete graph on V_t.

Set $H = \bigcup_{t \in T} K_t$. Then $(T, \mathcal{V})$ is also a tree-decomp. of H.

By Prop. 8.16 H is chordal and by the Corollary 8.13
$\omega(H) - 1 \leq tw(G)$. $\boxtimes$

8.6 Tree-width and forbidden minors

Some propositions without proofs.

Proposition 8.18 A graph has treewidth < 3 iff it has no K_4 as a minor.

Theorem 8.19 (Robertson and Seymour 1986)

Given a graph H, the graphs without an H minor have bounded treewidth iff H is planar.

Theorem 8.20 (Robertson and Seymour 1986)

$\forall$ natural number r, there exists a natural nr. k s.th. every graph of tree-width at least k has an $r \times r$ grid minor.

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