(7) Properties of almost all graphs

Recall: A graph property is a class of graphs which is closed under isomorphism, i.e. one which contains with every graph G also the graphs isomorphic to G.

Let $p\colon \mathbb{N} \to [0,1]$ be a fixed function (possibly constant) and let $\mathcal{P}$ be a graph property.
Consider $\lim_{n\to\infty} \mathbb{P}[G \in \mathcal{P}]$ for $G \in \mathcal{G}(n,p)$.

If $\lim_{n\to\infty} \mathbb{P}[G \in \mathcal{P}] = 1$, then $G \in \mathcal{P}$ **for almost all** $G \in \mathcal{G}(n,p)$**, or** $G \in \mathcal{P}$ **almost surely in** $\mathcal{G}(n,p)$.

If $\lim_{n\to\infty} \mathbb{P}[G \in \mathcal{P}] = 0$, then **almost no** $G \in \mathcal{G}(n,p)$ **has the property** $\mathcal{P}$**,** $G \notin \mathcal{P}$ **almost surely in** $\mathcal{G}(n,p)$.

Proposition 12.

For every constant $p \in (0,1)$ *and every given graph* H*, almost every* $G \in \mathcal{G}(n,p)$ *contains an induced copy of* H*.*

(7) Properties of almost all graphs (contd.)

Let $i, j \in \mathbb{N}$. $\mathcal{P}_{i,j}$ is the property that the considered graph G contains a vertex $v \in V(G) \setminus (U \dot{\cup} W)$ for every pair of disjoint vertex sets U, W ($U, W \subset V(G)$) with $|U| \leq i$, $|W| \leq j$, with the property that $\{v, u\} \in E(G)$, $\forall u \in U$, and $\{v, w\} \notin E(G)$, $\forall w \in W$.

Lemma 13.

For any constant $p \in (0, 1)$ and for all $i, j \in \mathbb{N}$ almost every $G \in \mathcal{G}(n, p)$ has the property $\mathcal{P}_{ij}$.

Corollary 14.

For any constant $p \in (0, 1)$ and for each $k \in \mathbb{N}$ almost every $G \in \mathcal{G}(n, p)$ is k-connected.

Proposition 15.

For any constant $p \in (0, 1)$ and for each $\epsilon \in \mathbb{R}$, $\epsilon > 0$, almost every $G \in \mathcal{G}(n, p)$ fulfills

$$\chi(G) Y \frac{log(1-q)}{2+\epsilon} \frac{n}{\log n}$$

where $q := 1 - p$.

(7) Threshold functions and second moments

Definition 3.

A function $t\colon \mathbb{N} \to (0, +\infty)$ *is called* **a threshold function for a graph property** $\mathcal{P}$ *if the following holds for every* $p\colon \mathbb{N} \to (0,1)$ *and* $G \in \mathcal{G}(n,p)$*:*

$$\lim_{n\to\infty} \mathbb{P}[G \in \mathcal{P}] = \begin{cases} 0 & \text{if } \lim_{n\to\infty} \dfrac{p(n)}{t(n)} = 0 \\ 1 & \text{if } \lim_{n\to\infty} \dfrac{p(n)}{t(n)} = \infty \end{cases} .$$

If $\mathcal{P}$ has a threshold function t, then for any positive constant c also ct is a threshold function for $\mathcal{P}$.

Definition 4.

$\mathcal{P}$ *is* **an increasing graph property** *if it closed under the addition of edges and vertices, i.e. if* $G \in \mathcal{P}$ *and* $G \subseteq H$ *imply* $H \in \mathcal{P}$*.*

Remark: Bollobás and Thomason (1987) have shown that all increasing graphs properties have threshold functions.

B. Bollobás and A.G. Thomason, Threshold functions, *Combinatorica* **7**, 1987, 35–38.

(7) Threshold functions and second moments (contd.)

For a considered graph property $\mathcal{P}$ introduce an apropriate random variable X. eg. the indicator random variable of $\mathcal{P}$ in $G(n,p)$, $X(G) \in \{0,1\}$, $X(G) = 1$ iff $G \in \mathcal{P}$. Then cast $\mathcal{P}$ by means of X, eg. $\mathcal{P} = \{G \colon X(G) = 1\}$.

In order to prove that t is a threshold function for $\mathcal{P}$ we show

(i) almost no $G \in \mathcal{G}(n,p)$ has the property $\mathcal{P}$ if p **is small as compared to** t, i.e. if $lim_{n\to\infty} p/t = 0$.

(ii) almost every $G \in \mathcal{G}(n,p)$ has the property $\mathcal{P}$ if p **is large as compared to** t, i.e. if $lim_{n\to\infty} p/t = +\infty$.

Ad (i) Compute an upper bound on $E(X)$, show that the bound tends to 0 as $n \to \infty$ and use Markov's inequality $\mathbb{P}[X \geq 1] \leq \mathbb{E}(x)$.

Ad (ii) In order to show that $\mathbb{P}[X \geq 1]$ is large, it is not enough to bound $\mathbb{E}(X)$ from below. Use Chebyshev's inequality.

Lemma 16.

(Chebyshev's Inequality)
For any $\lambda > 0$ and for any random variable X with expectation $\mathbb{E}(X) =: \mu$ and variance σ^2, the inequality $\mathbb{P}[|X - \mu| \geq \lambda] \leq \frac{\sigma^2}{\lambda^2}$ holds.

(7) Threshold functions and second moments (contd.)

Lemma 17.

Let X be a nonnegative random variable in $\mathcal{G}(n,p)$ with expectation $\mathbb{E}(X) =: \mu > 0$, for all large n, and variance σ^2 such that $\lim_{n\to\infty} \frac{\sigma^2}{\mu^2} = 0$. Then $X(G) > 0$ holds for almost all $G \in \mathcal{G}(n,p)$.

Let H be a graph with $k := |V(H)|$ and $l := |E(H)|$. Let $\mathcal{P}_H$ be the property of containing a subgraph isomorphic to H.

Let $X(G)$ be the number of subgraphs of G which are isomorphic to H for $G \in \mathcal{G}(n,p)$.

Let $\mathcal{H} := \{H' \colon H' \simeq H, V(H') \subseteq \{0,1,\ldots,n-1\}\}$. Then $E(X) = |\mathcal{H}|p^l \leq h\binom{n}{k}p^l$, where h is the number of graphs on k vertices $\{0,1,\ldots,k-1\}$ which are isomorphic to H.

Lemma 18.

If $t(n) = n^{-\frac{1}{\epsilon(H)}}$, where $\epsilon(H) = \frac{|E(H)|}{|V(H)|} = \frac{l}{k}$, and $p(n)$ fulfills $\lim_{n\to\infty} p(n)/t(n) = 0$, then almost no $G \in \mathcal{G}(n,p)$ lies in $\mathcal{P}_H$.

(7) Threshold functions and second moments (contd.)

Lemma 19.

If $t(n) = n^{-\frac{1}{\epsilon'(H)}}$ with $\epsilon'(H) := \max\{\epsilon(H') \colon H' \subseteq H\}$, and $p(n)$ fulfills $\lim_{n\to\infty} p(n)/t(n) = +\infty$, then almost every $G \in \mathcal{G}(n,p)$ lies in $\mathcal{P}_H$.

Theorem 20.

Let H be a graph with $|E(H)| \geq 1$. Then $t(n) = n^{-\frac{1}{\epsilon'(H)}}$ is a threshold function for $\mathcal{P}_H$.

Definition 5.

A graph H is called **balanced** *if $\epsilon'(H) = \epsilon(H)$ holds.*

Examples: trees, cycles.

Corollary 21.

If $k \in \mathbb{N}$, $k \geq 3$, then $t(n) = n^{-1}$ is a threshold function for the property of containing aa cycle on k vertices, for $G \in \mathcal{G}(n,p)$. ($t(n)$ does not depend on k!)

Corollary 22.

Let T be a tree with $V(T) =: k \geq 2$. Then $t(n) = n^{-\frac{k}{k-1}}$ is a threshold function for the property of containing an isomorphic copy of T, i.e. $G \in \mathcal{P}_T$, for $G \in \mathcal{G}(n,p)$.