(Chapter 7) Random graphs: definitions and elementary concepts

Let $V = \{0, 1, \ldots, n-1\}$ be a set of vertices, let $\mathcal{G}$ be the set of graphs G with vertex set $\mathcal{V}$, i.e. $V(G) = V$.

Goal: Turn $\mathcal{G}$ into a probability space!

Let $[V]^2 := \{\{i, j\} \colon i, j \in V, i \neq j\}$. Let $p \in [0, 1]$, $p \in \mathbb{R}$.

For each $e \in [V]^2$ decide by some random experiment whether $e \in E(G)$; the experiments are performed independently for each edge and the probability of success, i.e. accepting e as an edge, equals p.

Elementary events: a fixed graph G_0, $m := E(G_0)$,is generated by this series of random experiments with probability $p^m q^{\binom{n}{2}-m}$, where $q := 1 - p$.

Remark: The probability that the result of the series of experiments is a graph isomorphic to G is larger!

Theoretical question:: Existence of such a probability measure, $\mathcal{G}(n, p)$ on $\mathcal{G}$ for which every independent edge occurs independently with probability p.

(7) Random graphs: definitions and elementary concepts

Definition 1.

$\forall e \in [V]^2$ let $\Omega_e := \{0_e, 1_e\}$ be equiped with the probability measure $\mathbb{P}(\{1_e\}) = p$, $\mathbb{P}(\{0_e\}) = q = 1 - p$. The product probability space $\Omega := \prod_{e\in[V]^2} \Omega_e$ is denoted by $\mathcal{G}(n, p)$.
For any $\omega = (\omega_e)_{e\in[V]^2} \in \Omega$ we have $\omega_e \in \{1_e, 0_e\}$ for each $e \in [V]^2$. A graph G with $V(G) = V$ and $E(G) = \{e \in [V]^2 : \omega_e = 1\}$ is called a **random graph on V with edge probability** *p.*

Any set of graphs with vertex set V is an event in $\mathcal{G}(n, p)$.
Example: $A_e := \{\omega \in \Omega : \omega_e = 1\}$ is the set of graphs with $e \in E(G)$ or *the event that e is an edge of the random graph G.*

Proposition 1.

The events A_e, $e \in [V]^2$, are independent and occur with probability p.

Lemma 2.

For all inetgers n, k with $n \geq k \geq 2$, the probability that $G \in \mathcal{G}(n, p)$ contains a set of k independent vertices or a k-clique is at most

$$\mathbb{P}[\alpha(G) \geq k] \leq \binom{n}{k} q^{\binom{k}{2}} \text{ or } \mathbb{P}[\omega(G) \geq k] \leq \binom{n}{k} p^{\binom{k}{2}}, \text{ respectively.}$$

(7) Ramsey numbers

Theorem 3.

For every $r \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that every graph of order ta least n contains either K_r or $\bar{K}_r$ as an induced subgraph.

Definition 2.

For every $r \in \mathbb{N}$ the smallest $n \in \mathbb{N}$ such that every graph G with $|V(G)| \geq n$ contains either K_r or $\bar{K}_r$ as an induced subgraph is called the n-**th Ramsey number**, $\mathbf{R}(r)$.

Let H be a graph. The **Ramsey number** $\mathbf{R}(H)$ *is the smallest $n \in \mathbb{N}$ such that every graph G with $|V(G)| \geq n$ either G or $\bar{G}$ contains an induced subgraph isomorphic to H.*

Let H_1, H_2 be a pair of graphs. The **Ramsey number** $\mathbf{R}(H_1, H_2)$ *is the smallest $n \in \mathbb{N}$ such that for every graph G with $|V(G)| \geq n$ either G contains an induced subgraph isomorphic to H_1 or $\bar{G}$ contains an induced subgraph isomorphic to H_2. H.*

(7) Ramsey numbers (contd.)

Corollary 4.

For any $r \in \mathbb{N}$, $R(r) \leq 2^{2r-3}$ *holds.*

Proposition 5.

Let $s, t \in \mathbb{N}$ *be arbitrary natural numbers and* T *be a tree with* t *vertices. Then* $R(T, K_s) = (t-1)(s-1)+1$.

Theorem 6.

(Erdös 1947) For every integer k *with* $k \geq 3$, $R(k) \geq 2^{\frac{k}{2}}$ *holds.*

(7) Computing expectations

Let X be a graph invariant, e.g. the average degree, the connectivity, the girth, the chromatic number etc.

Then X is a random variable on $\mathcal{G}(n,p)$, $X\colon \mathcal{G}(n,p) \to \mathbb{R}$, $G \mapsto X(G)$.

The expecation of X is given as $\mathbb{E}(X) = \sum_{G\in\mathcal{G}(n,p)} \mathbb{P}(G)X(G)$. If $X(G) \in \mathbb{Z}_+$, for all $G \in \mathcal{G}(n,p)$, then

$$\mathbb{E}(X) = \sum_{k\in\mathbb{Z}_+} k\mathbb{P}(X(G) = k) = \sum_{k\in\mathbb{Z}_+} \mathbb{P}(X(G) \geq k)$$

Lemma 7.

(Markov's inequality)
Let $X \geq 0$ be a random variable on $\mathcal{G}(n,p)$ and $\alpha > 0$ a real number. Then $\mathbb{P}[X \geq \alpha] \leq \frac{\mathbb{E}(X)}{\alpha}$, provided that $\mathbb{E}(X)$ exists.

Lemma 8.

The expected number of cycle of length k in $\mathcal{G}(n,p)$ equals $\frac{(n)_k}{2k} p^k$, where $(n)_k := n(n-1)\dots(n-k+1)$ for all $n,k \in \mathbb{N}$, $3 \leq k \leq n$.

(7) The probabilistic method

Idea: In order to prove the existence of an object with a certain property consider a probabilistic space on the nonempty set of relevant objects and show that **the probability of existence of an object with the considered probybility in this space is positive**.

Theorem 9.

(Theorem of Erdös, 1959)
For every $k \in \mathbb{N}$, there exists a graph G with $g(G) > k$ and $\chi(G) > k$.

Lemma 10.

Let $k \in \mathbb{N}$ be arbitrary but fixed and let $p(n)$ be a function of n such that $p(n) \geq (6k \ln(n))n^{-1}$ holds for n large, i.e. for all $n \in \mathbb{N}$ with $n \geq n_0$, where n_0 is a certain fixed threshold. Then for $G \in \mathcal{G}(n, P)$ we have

$$\lim_{n \to \infty} \mathbb{P}\left[\alpha(G) \geq \frac{1}{2}\frac{n}{k}\right] = 0 \, .$$

Corollary 11.

There are graphs G with arbitrary large girth $g(G)$ and arbitrary large invariants $\kappa(G)$, $\epsilon(G)$ and $\delta(G)$.