Now let us consider the case where p is large as compared to t, i.e. $\gamma \xrightarrow[n\to\infty]{} \infty$.
Here we will work with another threshold function and then will easily convince ourselves that this other threshold function also works in Lemma 7.15.

Let $\varepsilon'(H) := \max\{\varepsilon(H') : H' \subseteq H\}$

"is a subgraph of"

Lemma 7.16 If $t = n^{-1/\varepsilon'(H)}$ and p is such that $\gamma = \frac{p}{t} \xrightarrow[n\to\infty]{} \infty$, then almost every $G \in \mathcal{G}(n,p)$ lies in $\mathcal{P}_H$.

Proof $\gamma = \frac{p}{t} \rightarrow p = \gamma \cdot t = \gamma\, n^{-1/\varepsilon'}$ with $\varepsilon' := \varepsilon'(H)$.

We will use the following inequality:

$\forall n \geq k$ $$\binom{n}{k} n^{-k} = \frac{1}{k!} \frac{n(n-1)\dots(n-k+1)}{n^k} =$$

$$= \frac{1}{k!} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \geq \frac{1}{k!} \left(\frac{n-k+1}{n}\right)^k$$

$$1 - \frac{k-1}{n} \geq 1 - \frac{k-1}{k}$$

$$\geq \frac{1}{k!} \left(1 - \frac{k-1}{k}\right)^k$$

Thus for k fixed and n large

$$\binom{n}{k} \geq \underbrace{\frac{1}{k!} \left(1 - \frac{k-1}{k}\right)^k}_{=: c_k} n^k = c_k\, n^k \quad (\cancel{****})$$

and c_k is independent on n!

Now we want to use Lemma 7.13, so we try to bound $\frac{\sigma^2}{\mu^2} = \frac{E(X^2)-\mu^2}{\mu^2}$ (with $\mu = E(X)$) from above. Rewrite X^2 as follows:

$$X^2(G) = |\underbrace{\{H \in \mathcal{H} : H \subseteq G\}}|^2 = \left|\{(H', H'') \in \mathcal{H}^2 : H' \subseteq G \text{ and } H'' \subseteq G\}\right|$$

Now compute $E(X^2)$ (by double counting):

$$E(X^2) = \sum_{(H',H'') \in \mathcal{H}^2} \mathbb{P}[H' \subseteq G \text{ and } H'' \subseteq G] =$$

$$= \sum_{(H',H'') \in \mathcal{H}^2} \mathbb{P}[H' \cup H'' \subseteq G]. \qquad (*)$$

Consider now $\mathbb{P}[H' \cup H'' \subseteq G] = p^{|E(H' \cup H'')|} = p^{2\ell - |E(H' \cap H'')|}$

because $E(H' \cup H'') = \underbrace{|E(H')|}_{=\ell} + \underbrace{|E(H'')|}_{=\ell} - |E(H' \cap H'')| = 2\ell - |E(H' \cap H'')|$ since $H', H'' \in \mathcal{H}$

But $|E(H' \cap H'')| = |V(H' \cap H'')| \cdot \varepsilon(H' \cap H'')$ $\leq i\, \varepsilon'$

$i := |H' \cap H''| = |V(H' \cap H'')|$

$H' \cap H''$ is a graph with vertex set $V(H') \cap V(H'')$ and edge set $E(H') \cap E(H'')$

This inequality is due to the fact that $H' \in \mathcal{H}$, $H'' \in \mathcal{H} \Rightarrow H' \cap H''$ ~~is~~ isomorphic to a subgraph of H, hence $\varepsilon(H' \cap H'') \leq \varepsilon'(H)$.

Thus we get

$$\mathbb{P}[H' \cup H'' \subseteq G] \leq p^{2\ell - \varepsilon' i} \qquad (***)$$

Rewrite the sum in (*):

$$E(X^2) = \sum_{(H',H'')\in\mathcal{H}^2} \mathbb{P}[H'\cup H''\subseteq G] = \sum_{i=0}^{k} \sum_{\substack{(H',H'')\in\mathcal{H}^2\\ |H'\cap H''|=i}} \mathbb{P}[H'\cup H''\subseteq G]$$

$(H',H'')\in\mathcal{H}_i^2$ (**)

Let $\mathcal{H}_i^2 := \{ (H',H'')\in\mathcal{H}^2 : |H'\cap H''| = i\}$, $i\in\overline{0,k}$

and $A_i := \sum_{(H',H'')\in\mathcal{H}_i^2} \mathbb{P}[H'\cup H''\subseteq G]$.

So (**) becomes $E(X^2) = \sum_{i=0}^{k} A_i$ (~~***~~)

If $i=0$: $|H'\cap H''|=0 \Rightarrow V(H')\cap V(H'') = \emptyset \Rightarrow$

the events $H'\subseteq G$, $H''\subseteq G$ are independent $\Rightarrow$

$$A_0 = \sum_{(H',H'')\in\mathcal{H}_0^2} \mathbb{P}[H'\cup H''\subseteq G] = \sum_{(H',H'')\in\mathcal{H}_0^2} \underbrace{\mathbb{P}[H'\subseteq G]}_{p^\ell}\,\underbrace{\mathbb{P}[H''\subseteq G]}_{p^\ell}$$

$$\cancel{\leq}\ \sum_{(H',H'')\in\mathcal{H}^2} \mathbb{P}[H'\subseteq G]\,\mathbb{P}[H''\subseteq G] =$$

$$= \underbrace{\sum_{H'\in\mathcal{H}} \mathbb{P}[H'\subseteq G]}_{E(X)=\mu}\ \underbrace{\sum_{H''\in\mathcal{H}} \mathbb{P}[H''\subseteq G]}_{E(X)=\mu} = \mu^2$$

Next we estimate A_i for $i\geq 1$.

Rewrite $A_i = \sum_{H',H''\in\mathcal{H}_i^2} \mathbb{P}[H'\cup H''\subseteq G] = \sum_{H'\in\mathcal{H}} \sum_{\substack{H''\in\mathcal{H}\\ |H'\cap H''|=i}} \mathbb{P}[H'\cup H''\subseteq G]$

H'' 0. x ... x . x - x $k-1$ $H \cong X$

For a fixed H' the second sum has $\binom{k}{i}\binom{n-k}{k-i}h$ summands $=$ # graphs $H''\in\mathcal{H}$ with $|V(H'\cap H'')|=i$

So $\forall i \geq 1$ we get

$$A_i = \sum_{H' \in \mathcal{H}} \sum_{\substack{H'' \in \mathcal{H}: \\ |H' \cap H''| = i}} \mathbb{P}[H' \cup H'' \subseteq G] \overset{(***)}{\leq} \sum_{H' \in \mathcal{H}} \binom{k}{i}\binom{n-k}{k-i} h\, p^{2e - i\varepsilon'}$$

$p^{2e - i\varepsilon'}$ $p^{2e - i\varepsilon'}$ p p

$$= |\mathcal{H}| \binom{k}{i}\binom{n-k}{k-i} h\, p^{2e} \left(\gamma n^{-1/\varepsilon'}\right)^{-i\varepsilon'} \leq$$

$p = \gamma t$

$\binom{k}{i}\binom{n-k}{k-i} = \binom{k}{i}\frac{(n-k)_{k-i}}{(k-i)!}$

$$\leq \underbrace{|\mathcal{H}|\, p^e}_{=\mu = E(X)} \, c_1\, n^{k-i}\, h\, p^e \gamma^{-i\varepsilon'} n^i$$

with $c_1 = \binom{k}{i}\frac{1}{(k-i)!}$

and $(n-k)_{k-i} \leq (n-k)^{k-i} \leq n^{k-i}$

Observe: c_1 independent on n!

Also c_2 is independent on n since c_k is independent on n.

$$= \mu\, c_1\, n^k h\, p^e \gamma^{-i\varepsilon'} \leq$$

$$\overset{(****)}{\leq} \mu\, c_2 \underbrace{\underbrace{\binom{n}{k} h}_{=|\mathcal{H}|} p^e}_{=\mu} \gamma^{-i\varepsilon'} \text{ with } c_2 = c_1/c_k$$

$$= \mu^2 c_2 \gamma^{-i\varepsilon'} \underset{(\text{if } \gamma \geq 1)}{\leq} \mu^2 c_2 \gamma^{-\varepsilon'}$$

Summarizing we obtain from $(*****)$

$$E(X^2) = \sum_{i=0}^{k} A_i = A_0 + \sum_{i=1}^{k} A_i \leq \mu^2 + \sum_{i=1}^{k} \mu^2 c_2 \gamma^{-\varepsilon'}$$

and

$$\frac{\sigma^2}{\mu^2} = \frac{E(X^2) - \mu^2}{\mu^2} \leq \sum_{i=1}^{k} c_2 \gamma^{-\varepsilon'} = \underbrace{k\, c_2}_{=: c_3} \gamma^{-\varepsilon'} = c_3 \gamma^{-\varepsilon'}$$

indep on n!

Finally $\frac{\sigma^2}{\mu^2} \leq c_3 \gamma^{-\varepsilon'} \xrightarrow[\gamma \to \infty]{} 0$ (because $\varepsilon' > 0$)

and lemma 7.14 implies $\mathbb{P}[X = 0] = \mathbb{P}[H \not\subseteq G] \xrightarrow[n \to \infty]{} 0$

$\Rightarrow$ $\mathbb{P}[X > 0] \xrightarrow[n \to \infty]{} 1$

Theorem 7.17 (Erdös, Rényi 1960, Bollobas 1981)

Let H be a graph with $|E(H)| \geq 1$.
Then $t = n^{-1/\varepsilon'(H)}$ is a threshold function for $\mathcal{P}_H$.

Proof We have to show:

(1) $\mathbb{P}[H \subseteq G] \xrightarrow[n\to\infty]{} 0$ if p is such that $\gamma \xrightarrow[n\to\infty]{} 0$

(2) $\mathbb{P}[H \subseteq G] \xrightarrow[n\to\infty]{} 1$ if p is such that $\gamma \xrightarrow[n\to\infty]{} \infty$

(2) is exactly what is shown in Lemma 7.16

To show (1) consider $H' \subseteq H$ s.th. $\varepsilon(H') = \varepsilon'(H)$

and apply Lemma 7.15 for H':

$$\mathbb{P}[H' \subseteq G] \xrightarrow[n\to\infty]{} 0 \quad \text{if } p \text{ is such that } \gamma \xrightarrow[n\to\infty]{} 0 \text{ with } \gamma = \frac{p}{t} = \frac{p}{n^{-1/\varepsilon(H')}}$$

Finally observe that
$$\mathbb{P}[H \subseteq G] \leq \mathbb{P}[H' \subseteq G] \text{ (because } H' \subseteq H) \xrightarrow[n\to\infty]{} 0$$
and this completes the proof. $\square$

Remark: Theorem 7.17 is particularly easy to apply if H is such that $\varepsilon'(H) = \varepsilon(H)$.
Such graphs are called balanced graphs.

Examples of balanced graphs: trees, cycles

$t = n^{-1/\varepsilon'}$

Corollary 7.18 If $k \geq 3$ then $t = n^{-1}$ is a threshold function for the property of containing a k-cycle.

$p = o(\frac{1}{n}) \Longrightarrow C_k \not\subseteq G$ a.s. in $G(n,p)$

$p = n^{-1+\varepsilon} \Longrightarrow C_k \subseteq G$ a.s. in $G(n,p)$

Proof: Observe that for every cycle C_k $\varepsilon'(C_k) = 1$ holds. Indeed every proper subgraph of a cycle is a collection of paths (may be of length 0, i.e. have 1 vertex and zero edges) And the density of each path is ≤ 1 whereas $\varepsilon(C_k) = 1$. Thus $\varepsilon'(C_k) = 1$.

According to Th. 7.17 the threshold function for $\mathcal{P}_{C_k}$ is $n^{-1/\varepsilon'(C_k)} = \frac{1}{n}$

Remark: $t(n) = \frac{1}{n}$ does not depend on k!

Corollary 7.19 If T is a tree with $|V(T)| = k \geq 2$ then $t(n) = n^{-k/(k-1)}$ is a threshold function for the property of containing an isomorphic copy of T, i.e $T \subseteq G$ or $G \in \mathcal{P}_T$.

Proof Observe that $\varepsilon(T) = \frac{k-1}{k}$ if $|V(T)| = k$.

Each subgraph of T is a forest on k or less vertices; the density of such a subgraph is $\leq \frac{k-1}{k}$.

Therefore $\varepsilon'(T) = \varepsilon(T) = \frac{k-1}{k}$ and the corollary follows then directly from Theorem 7.17.

$-1/\varepsilon' = -\frac{k}{k-1}$