Proof of Theorem 7.8

Assume w.l.o.g. $k \geq 3$ ($g(G) \geq 3$ holds for any graph).
Fix $0 < \varepsilon < \frac{1}{k}$ and set $p := n^{\varepsilon - 1}$.

Let $X(G)$ denote the number of short cycles in a random graph $G \in \mathcal{G}(n,p)$ (i.e. the number of cycles of length at most k).

By Lemma 7.7 (on the exp. $E(X)$ of X) we have

$$\underline{E(X)} = \sum_{i=3}^{k} \underbrace{\frac{(n)_i}{2i} p^i}_{= E(X_i)} \leq \frac{1}{2} \sum_{i=3}^{k} n^i p^i \overset{(*)}{\leq} \underline{\frac{1}{2}(k-2) n^k p^k}$$

$= E(X_i)$ where X_i is the nr of i-cycles

The inequality $(*)$ holds because $n \cdot p = n n^{\varepsilon - 1} = n^{\varepsilon} \geq 1$ and therefore $(np)^i \leq (np)^k$, $\forall i \leq k$.

Now by Lemma 7.6 (Markov's inequality) we have

$$\mathbb{P}\left(X \geq \frac{n}{2}\right) \leq \frac{E(X)}{\frac{n}{2}} \leq \frac{(k-2)\, n^k p^k}{n} = (k-2)\, n^{k-1} p^k$$

$$= (k-2)\, n^{k-1}\, n^{(\varepsilon - 1)k} = (k-2)\, n^{k\varepsilon - 1}$$

By our choice of $\varepsilon \in (0, \frac{1}{k})$ we have $\varepsilon k - 1 < 0$

Thus $\underline{\lim_{n \to \infty} \mathbb{P}\left(X \geq \frac{n}{2}\right) = 0}$

<u>Let now n be large enough</u> such that $\underline{\mathbb{P}\left(X \geq \frac{n}{2}\right) < \frac{1}{2}}$ and $\underline{\mathbb{P}\left(\alpha \geq \frac{1}{2} n/k\right) < \frac{1}{2}}$ (the latter is possible by our choice of p in Lemma 7.7; there we had $p \geq \frac{6k \ln n}{n}$ which is obviously fulfilled by $p = \frac{n^{\varepsilon}}{n}$ $\forall \varepsilon > 0$, if n is large enough)

Then there exists a graph $G \in \mathcal{G}(n,p)$ with fewer than $\frac{n}{2}$ short cycles and $\alpha(G) < \frac{1}{2}\frac{n}{k}$.

From each of those cycles delete a vertex and let H be the graph resulting after this process.

Then $|H| \geq \frac{n}{2}$ and H has no short cycles, so $g(H) > k$.

By the definition of G we get

$$\chi(H) \geq \frac{|H|}{\alpha(H)} \geq \frac{n/2}{\alpha(G)} > k$$ and this completes the proof. ∎

Corollary: There are graphs with arbitrary large girth and arbitrary large values of the invariants κ, ε and δ.

Proof Sketch, details to be worked out as a homework or exercise! We use a fact derived as a corollary of

$$\chi(G) \leq \operatorname{col}(G) = \max\{\delta(H) : H \subseteq G\} + 1 \text{ (Chapter 5)}$$

Corollary: Every graph G has a subgraph of minimum degree at least $\chi(G) - 1$.

This guarantees that $\delta(G)$ is large.

Further we use the Theorem of Mader

Th: Let $0 \neq k \in \mathbb{N}$. Every graph with $d(G) \geq 4k$ has a $(k+1)$-connected subgraph H such that

$$\varepsilon(H) > \varepsilon(G) - k.$$

(Recall: $\varepsilon(G) := \frac{|E(G)|}{|V(G)|}$, the density of the graph

$$d(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \deg(v)$$

κ the connectivity number, i.e. the smallest number k s.th. removing $k-1$ vertices does not disconnect G with $V(G) \geq k$.

Properties of almost all graphs

Recall: a graph property is a class of graphs which is closed under isomorphism, i.e. one that contains with every graph G also the graphs isomorphic to G.

If $p = p(n)$ is a fixed function (possibly constant) and $\mathcal{P}$ is a graph property we may consider $\lim_{n\to\infty} \mathbb{P}(G \in \mathcal{P})$ for $G \in \mathcal{G}(n,p)$.

If $\lim_{n\to\infty} \mathbb{P}(G \in \mathcal{P}) = 1$ we say $G \in \mathcal{P}$ for almost all $G \in \mathcal{G}(n,p)$, or that $G \in \mathcal{P}$ almost surely in $\mathcal{G}(n,p)$.

If $\lim_{n\to\infty} \mathbb{P}(G \in \mathcal{P}) = 0$ we say that almost no $G \in \mathcal{G}(n,p)$ has property $\mathcal{P}$ (example: we proved in Lemma 7.8 that for a certain p, almost no $G \in \mathcal{G}(n,p)$ has a set of more than $\frac{1}{2}\frac{n}{k}$ independent vertices) $p \geq \frac{6k\ln n}{n}$

To illustrate this new concept we prove that for constant p, every fixed abstract graph H is an induced subgraph of almost all graphs.

Proposition 7.10 For every constant $p \in (0,1)$ and every graph H, almost every $G \in \mathcal{G}(n,p)$ contains an induced copy of G.

Proof Let H be given and $k := |H|$. $|H| = k$
If $n \geq k$ and $U \subseteq \{0,\dots,n-1\}$ with $|U| = k$ is a fixed set of k vertices of G then $G[U]$ is isomorphic to H

Perfect graphs 18.

with a certain probability $r > 0$. This probability depends on p but not on n (why?) $r \ll 0$ & $r < 1$ ok nothing to show

G contains a collection of $\lfloor \frac{n}{k} \rfloor$ disjoint such sets U. The probability that none of the $G[U]$ is isomorphic to H is $(1-r)^{\lfloor \frac{n}{k} \rfloor}$ because these events are independent, because the sets U are disjoint and hence also the edge sets in $[U]^2$ are disjoint.

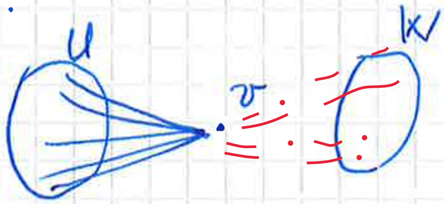
Thus $\mathbb{P}(H \not\subseteq G \text{ induced}) \le (1-r)^{\lfloor \frac{n}{k} \rfloor} \xrightarrow[n\to\infty]{} 0$

prob that $G[U] \not\cong H$ for any chosen set U, among $U_1, \dots U_{\lfloor \frac{n}{k} \rfloor}$

(r is fixed and also k)

The following lemma is a technical result needed to prove that many properties are almost sure in $\mathcal{G}(n,p)$ for some suitably chosen p.

Given $i, j \in \mathbb{N}$, let $P_{i,j}$ be the property that the considered graph contains, for any disjoint vertex sets U, W with $|U| \le i$ and $|W| \le j$ a vertex $v \notin U \dot\cup W$ that is adjacent to all vertices in U but none in W.



Lemma 7.10 $\forall p \in (0,1)$ and $i, j \in \mathbb{N}$ almost every graph $G \in \mathcal{G}(n,p)$ has the property $P_{i,j}$.

Proof For fixed U, W and $v \in G - (U \dot\cup W)$, the prob. that v is adjacent to all $u \in U$ but to no $w \in W$ is

$q := 1-p$ $\qquad p^{|U|} q^{|W|} \ge p^i q^j$

Thus the probability that no such a suitable v exist for these sets U and $|X|$ is

$$\left(1-p^{|U|}q^{|W|}\right)^{n-|U|-|W|} \le \left(1-p^{i}q^{j}\right)^{n-i-j} \quad (\text{for } n \ge i+j),$$

$\le \left(1-p^{i}q^{j}\right)^{n-|U|-|X|} \le$

since the corresponding events are independent for different v.

length$\left(\underbrace{\cdots}_{U}\ \underbrace{\cdots}_{|X|}\right) \le i+j$

There are at most n^{i+j} such sets U, W in $V(G)$.

Thus the probability that some such pair has no suitable v is at most

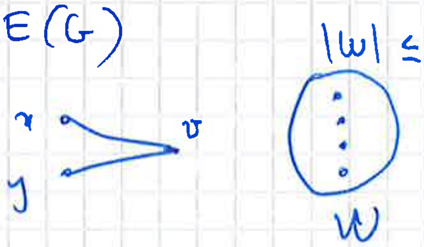
$$n^{(i+j)}\left(1-p^{i}q^{j}\right)^{n-i-j} \xrightarrow[n\to\infty]{} 0$$

$E_{U,W}$ event that no v with ... exists

$$\mathbb{P}\Big(\bigcup_{|U|\le i,\ |W|\le j,\ U\cap W=\emptyset} E_{U,W}\Big)$$

Corollary: $\forall p\in(0,1)$, $\forall k\in\mathbb{N}$, almost every $G\in\mathcal{G}(n,p)$ is k-connected.

Proof: By Lemma 7.10 above it is enough to show that every graph in $\mathcal{P}_{2,k-1}$ is k-connected.
This is simple: Every graph G in $\mathcal{P}_{2,k-1}$ is of order at least $k+2$ and $\forall W\subseteq V(G)$ with $|W|\le k-1$, $\forall x,y\notin W$, $\exists v\notin W\cup\{x,y\}$ with $\{v,x\}\in E(G)$, $\{v,y\}\in E(G)$.



Thus W does not separate x from y.
Since this holds $\forall W\subseteq V(G)$ with $|W|\le k-1$ we conclude that G is k-connected. □

Remark: The x-y-paths constructed in the proof of the Corollary have length 2, which is a stronger result than needed for the k-connectivity!