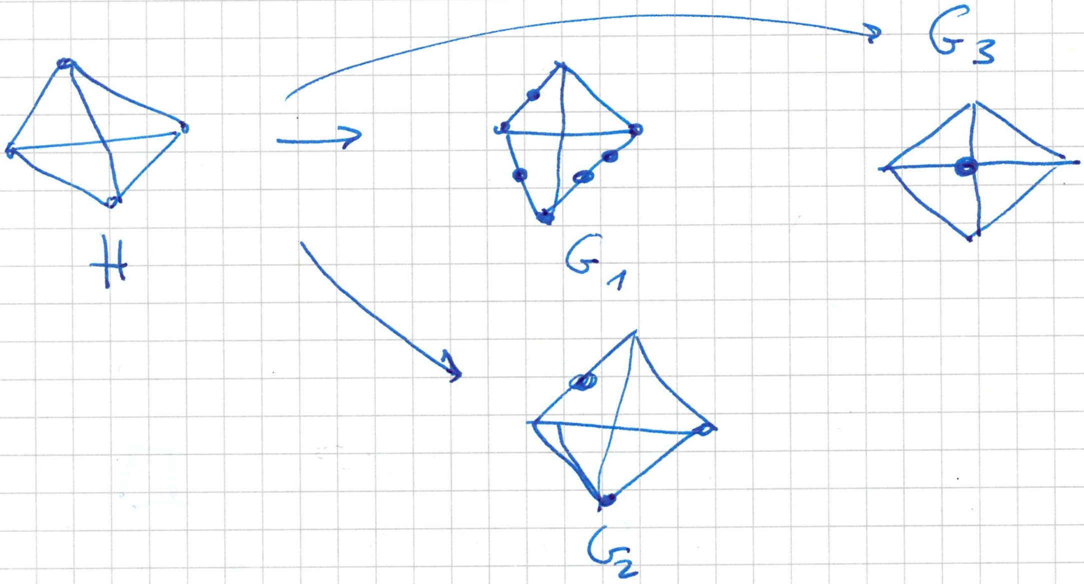
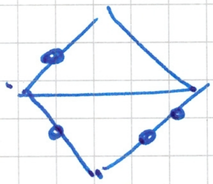
Examples of topological subgraphs



G_1, G_2 are subdivision (topological subgraphs of H)

G_1 and G_2 here homeomorphic

G_3 is not a subdivision of H



G_1

contains

K_3

as a subdivision (because it contains a subdivision of K_3, namely

as a subgraph)

Proof of Lemma 6

Let G be a graph and $e \in E(G)$, $e = \{x, y\}$.
Assume G/e contains a subdivision S of K_5 or $K_{3,3}$. Let z be the node representing e in G/e.
Assume w.l.o.g. S contains vertex z.
$\forall$ edge $\{z, v\}$ in G/e there is at least one of $\{x, v\}$, $\{y, v\}$ in $E(G)$. Construct $T \subseteq E(G)$ as the union of $E_1 := \{e \in E(G) : e \cap \{z\} = \emptyset\}$ and exactly one edge $\{x, u\}$ or $\{y, u\}$, $\forall$ $\{z, u\} \in S$

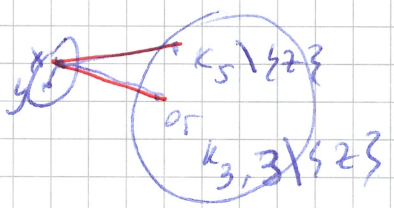
We distinguish some cases

a) $\deg_S(z) = 2$

Then z is a subdivision vertex in S

a1) $\deg_T(x) = 2$
$\deg_T(y) = 0$

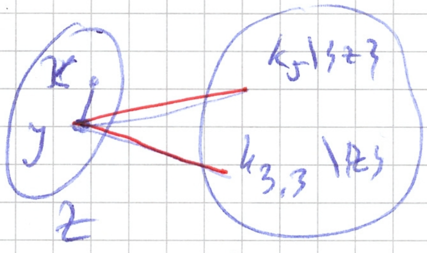
with $\{x, u_1\} \in T$
$\{x, u_2\} \in T$
$E_1 \cup \{\{x, u_1\}, \{x, u_2\}\}$ is a subdivision of $K_{3,3}$ or K_5 in G



x is not an "original" vertex of K_5 or $K_{3,3}$

a2) $\deg_T(y) = 2$
$\deg_T(x) = 0$

with $\{y, u_1\} \in T$
$\{y, u_2\} \in T$
Then $E_1 \cup \{\{y, u_1\}, \{y, u_2\}\}$ is a subdivision of $K_{3,3}$ or K_5 in G

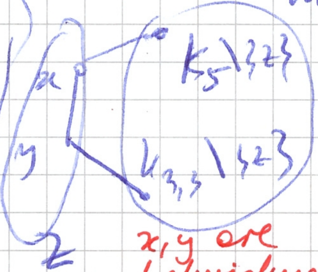


y is not an original vertex of K_5 or $K_{3,3}$

a3) $\deg_T(x) = 1$ and $\deg_T(y) = 1$

with $\{x, u_1\} \in T$
$\{y, u_2\} \in T$

Then $E_1 \cup \{\{x, u_1\}, \{y, u_2\}, \{x, y\}\}$ is a subdivision of $K_{3,3}$ or K_5 in G

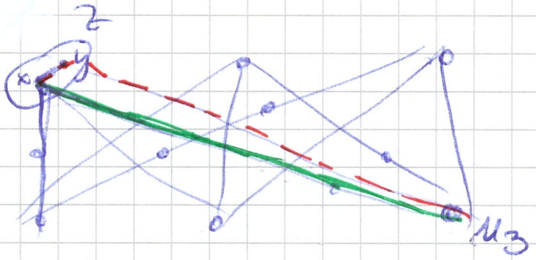


x, y are subdividing vertices

b) $\deg_S(z) = 3$

S is a subdivision of $K_{3,3}$
and z is on of the original vertices of $K_{3,3}$

b1) $\deg_T(x) = 2$ and $\deg_T(y) = 1$ (or $\deg_T(y) = 2$ and $\deg_T(x) = 1$)



y is a subdividing vertex in T instead of edge $\{z, u_3\} \in S$

Then $E_1 \cup \{\{x, u_1\}, \{x, u_2\}, \{x, y\}, \{y, u_3\}\}$ is a subdivision of $K_{3,3}$ in G

b2) $\deg_T(x) = 3$ $\deg_T(y) = 0$ (or $\deg_T(y) = 3$ $\deg_T(x) = 0$)

trivial : "replace" z by x (or y) in S'

and obtain T as a subdivision of $K_{3,3}$ in G

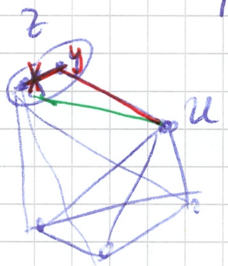
c) $\deg_S(z) = 4$ $\rightarrow$ z is an "original" vertex of K_5
(not a subdividing vertex)

c1) $\deg_T(x) = 4$ and $\deg_T(y) = 0$ (or $\deg_S(x) = 0$ and $\deg_T(y) = 4$)

trivial, "replace z by x (or y) in S"

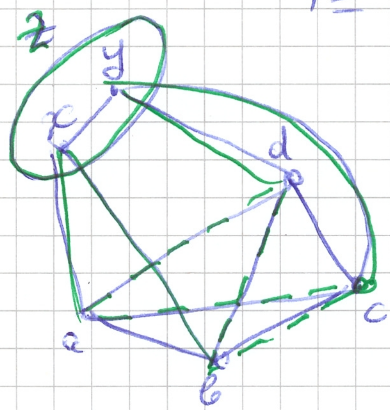
c2) $\deg_T(x) = 3$ and $\deg_T(y) = 1$ (or $\deg_T(y) = 3$ and $\deg_T(x) = 1$)

replace $\{z, u\}$ in S
by (x, y, u) in T



c3) $\deg_T(x)=2$ and $\deg_T(y)=2$

$T = \{\{x,a\},\{x,b\},\{x,y\},\{y,c\},\{y,d\}\}$
$\cup E_1$ } edges of K_4 on $\{a,b,c,d\}$



contains a subdivision of $K_{3,3}$ with partition $\{x,d,c\} \cup \{y,a,b\}$

consisting of the green edges dashed and continued

$\square$

Proof of the theorem of Kuratowski

$\Rightarrow$ The direction

G planar $\Rightarrow$ G does not contain a subdivision of K_5 or $K_{3,3}$ is trivial.

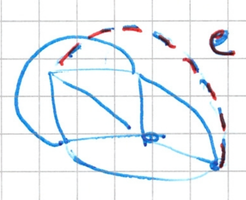
$\Leftarrow$ Assume G does not contain a subdivision of K_5 or $K_{3,3}$

Show G is planar.

Induction by $|V|$.

Observe that for $|V| \leq 5$ the claim holds.

(indeed $K_5 \setminus e$ is planar



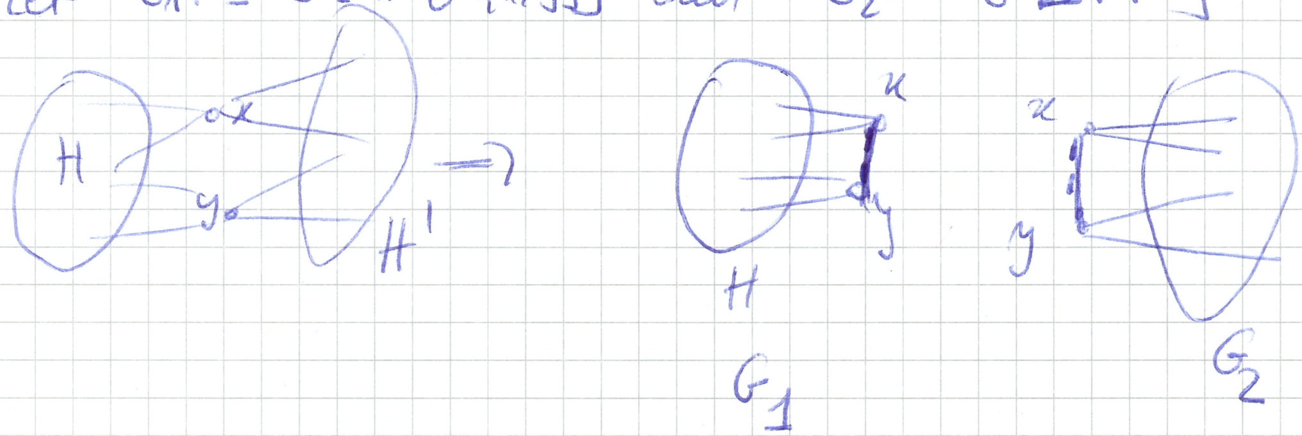
)

Thus, assume $|V(G)| \geq 6$ and G contains no subdivision of K_5 or $K_{3,3}$

Case I G is not 3-connected

Since G planar $\Leftrightarrow$ every block is planar assume that G is 2-connected. Let $\{x,y\}$ be a separator.

Let H be a connected component of $G \setminus \{x,y\}$
Let $G_1 := G[H \cup \{x,y\}]$ and $G_2 := G[V \setminus H]$



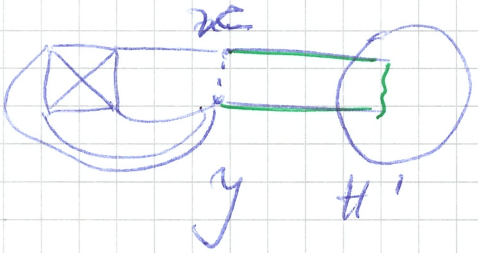
If $\{x,y\} \notin E(G)$, we add to G_1 and G_2

Since G is 2-connected, each vertex in $\{x,y\}$ is connected to all components of $G \setminus \{x,y\}$ (otherwise we would have a separator with cardinality 1 in G)

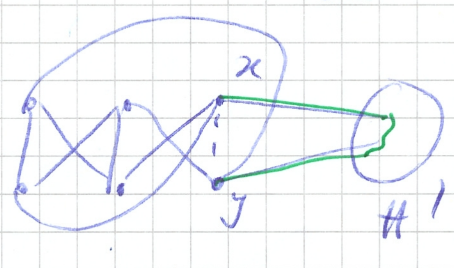
Observe that $G_1 + \{x,y\}$ does not contain a subd. of K_5 or $K_{3,3}$.

And analogously also $G_2 \cup \{x,y\}$ does not have such a subdivision.

Indeed if $G_1 + \{x,y\}$ would have such a subdiv. we could replace $\{x,y\}$ by the path $xH'y$



K_5



$K_{3,3}$

So $G_1 + \{x,y\}$, $G_2 + \{x,y\}$ do not contain subdivisions of K_5 or $K_{3,3}$.

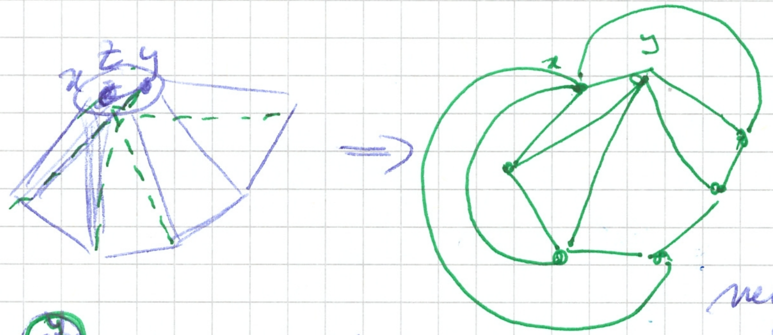
By induction they are planar.

Embed $G_1 + \{x, y\}$ and $G_2 + \{x, y\}$ in the plane such that $\{x, y\}$ lies in the outer-face, respectively, and merge these 2 embeddings in $\{x, y\}$ to obtain an embedding of G.

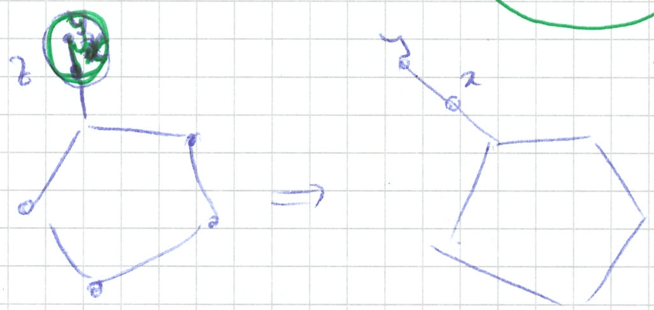
Case 2 G is 3-connected

From a lemma of Chapter 3: $\exists e = \{x y\} \in E(G)$ such that G/e is 3-connected. Let z be the vertex replacing e in G/e. Due to lemma 6 G/e contains no subdivision of $K_{3,3}$ or K_5. Thus G/e is planar by induction.

If there is a planar embedding of G/e containing z in the border of the outer-face this embedding can be extended to a planar embedding of G



Draw neighbors of y inside the outer-face and the neighbors of x outside the outer-face



Thus assume w.l.o.g that z lies in the ~~interior~~ complement of the union of the outer face and its border.

Embed $G_1+\{x,y\}$ and $G_2\cup\{x,y\}$ in the plane such that $\{x,y\}$ lies in the outer-face, respectively, and merge these two embeddings in $\{x,y\}$ to obtain an embedding of G. 7

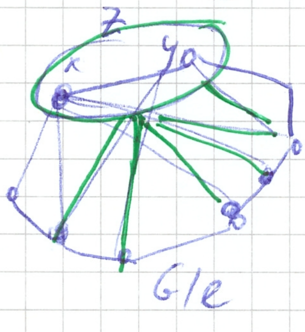
Case 2 G is 3-connected

From a lemma of Chapter 3 we know: $\exists\ e=\{x,y\}\in E(G)$ such that G/e is 3-connected. Let z be the vertex replacing e in G/e

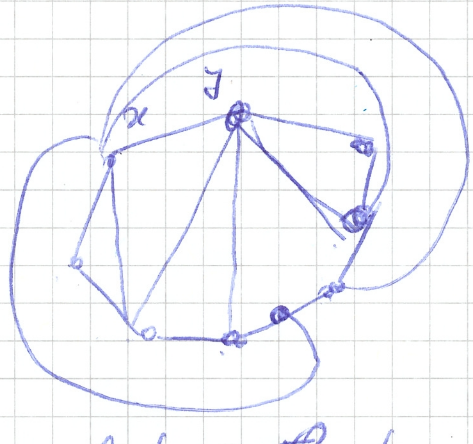
Due to lemma 6 G/e contains no subdivision of $K_{3,3}$ or K_5

Thus G/e is planar by induction

Consider a planar embedding of G/e in the plane and assume w.l.o.g that z lies in the border of face F. which is not the outer face. (If z was a vertex on the border of the outerface and the planar embedding of G/e could be extended to a planar embedding of G.)

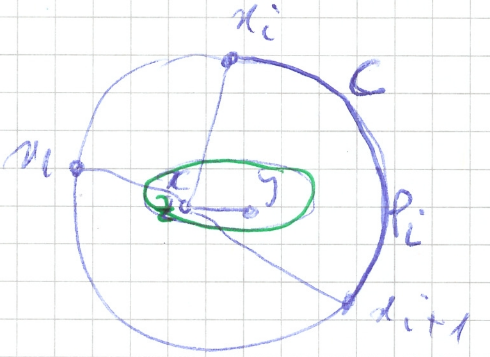


$\Longrightarrow$



G/e 3-connected $\Rightarrow$ G/e 2-connected $\rightarrow$ Border of F is a cycle C

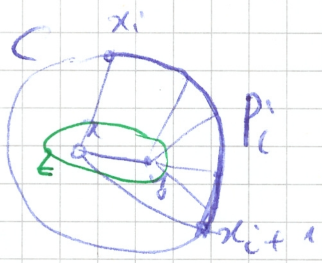
Let C be the cycle obtained as the border of the union of faces which contain z on their borders.



Let $x_1, \dots x_k$ be the neighbors of x in C, for example in the clockwise order, and let P_i be the $x_i - x_{i+1}$ - subpath of C (set $x_{k+1} \equiv x_1$).

Distinguish the following cases

Case 2a: $\exists\, i \in \{1, 2, \dots, k\}$ such that $N_G(y) \setminus \{x\} \subseteq V(P_i)$

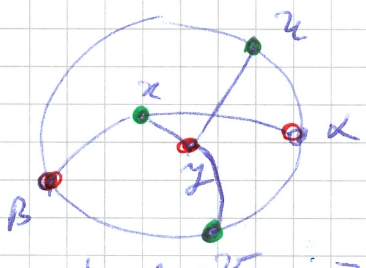


Then the planar embedding of G/e can be trivially extended to an embedding of G.

Case 2b $\forall\, i \in \{1, 2, \dots k\}$ $\exists$ neighbor of y in $V(P_i)$ and $\exists$ neighbor of y not in $V(P_i)$.

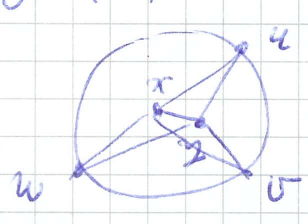
Case 2b(i): $N_G(y) \setminus \{x\} \subseteq \{x_1, \dots, x_k\}$

If $\deg_G(y) = 3$, then let $\{y, u\} \in E(G)$ and $\{y, v\} \in E(G)$.



Since Case 2a does not hold $\exists\, \alpha, \beta \in N_G(x) \setminus \{y\}$ as in the picture. Then we have a subdivision of $K_{3,3}$ in G (with red-green partition) ↯

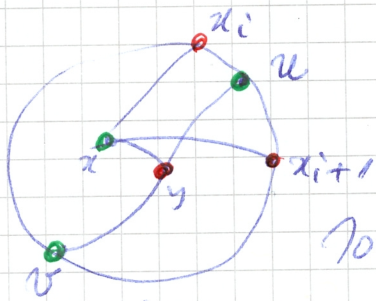
If $\deg_G(y) \geq 4$ then let u, v, w be in $N_G(y) \cap \{x_1, \dots, x_k\}$.



Thus we have a subdivision of K_5 in G ↯

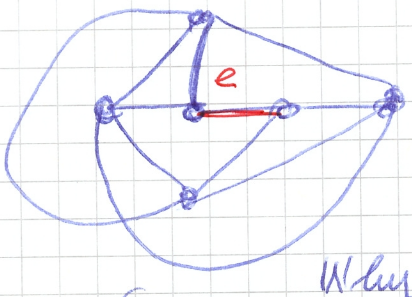
Case 2 b(ii): $N_G(y) \setminus \{x\} \not\subseteq \{x_1, \dots, x_k\}$

$\Rightarrow \exists$ neighbor u of y which is an inner point of some P_i and $\exists$ neighbor v of y which is not in $V(P_i)$. Then we have a subdivision of $K_{3,3}$ in G, with the red-green partition



So in all cases which can arise we can extend the planar embedding of G/e to a planar embedding of G. $\boxed{\Sigma}$

Example of G containing H as a minor but not as a subdivision:



G

G contains K_5 as a minor (contract e) but not as a subdivision

Why not as a subdivision?

We would need 5 vertices of degree 4!

Proof of Th. 7 will be done at the exercises

Proof of Corollary 8

~~...~~ "$\Rightarrow$)" G is planar $\Rightarrow$ every minor of it is planar
$\Rightarrow K_5$ and $K_{3,3}$ are not minors of G
(trivial)

$\Leftarrow$ Assume G does not contain K_5 or $K_{3,3}$ as a minor.

$\Rightarrow$ K_5 or $K_{3,3}$ are also not contained as subdivisions (containment as subdivision implies containment as minor)

$\xRightarrow{\text{Th. Kuratowski}}$ G is planar $\blacksquare$