Proof of Th. 9

$\Rightarrow$ Let $a_1, a_2, \dots, a_n$ with $0 < a_1 \le a_2 \le \dots \le a_n < n$ be an integer sequence with $n \ge 3$.

Assume that $a_i \le i \Rightarrow a_{n-i} \ge n-i$, $\forall\ i < \frac{n}{2}$ $(*)$

and show that $a_1, \dots, a_n$ is Hamiltonian

Via contradiction: $\exists$ graph G with degree sequence

$d_1 \le d_2 \le \dots \le d_n$ such that $d_i \ge a_i\ \forall i \in \overline{1,n}$ (1)

and G is not Hamiltonian

Choose G to have the max. number of edges $(**)$

The following holds

$\forall i < \frac{n}{2}$ $\quad d_i \le i \Rightarrow a_i \le d_i \le i \Rightarrow a_{n-i} \ge n-i \Rightarrow d_{n-i} \ge a_{n-i} \ge n-i$ (2)

Consider a pair $x, y \in V(G)$, $x \ne y$ with $\deg(x) \le \deg(y)$, $\{x,y\} \notin E(G)$ and $\deg(x) + \deg(y)$ as large as possible

The degree sequence of $G + \{x,y\}$ is pointwise larger than that of G.

By the maximality of G $(**)$ we get:

$G + \{x,y\}$ is Hamilt. and every H.C. in $G + \{x,y\}$ contains e.

Let H be such a HC in $G + \{x,y\}$.

Then $H - \{x,y\}$ is a Hamilt. path P in G

Let $P = (x_1, \dots, x_n)$ with $x_1 = x$ and $x_n = y$ (or $x_1 = y$ and $x_n = x$)

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As in the proof of Dirac's theorem, consider

$I := \{ 1 \le i \le n-1 : \{x, x_{i+1}\} \in E(G) \}$

$J := \{ 1 \le j \le n-1 : \{y, x_j\} \in E(G) \}$

Then $I \cup J \subseteq \{1, 2, \ldots, n-1\}$ and $I \cap J = \emptyset$ (otherwise we would have an HC in G)

Hence $\deg(x) + \deg(y) = |I| + |J| \le n-1 < n$ (3)

Thus $\deg(x) < \frac{n}{2}$ (as the smaller among $\deg(x), \deg(y)$)

Since $\{x_i, y\} \notin E(G)$, $\forall i \in I$ and $\{x, y\}$ was chosen to maximize $\deg(x) + \deg(y)$ we get

$\deg(x_i) \le \deg(x) \quad \forall i \in I$. Let $h := \deg(x)$

Thus G has at least $|I| = h$ vertices with degree at most h

$\Longrightarrow$ thus the h-th number d_h in the degree sequence of G is $\le h$, so $d_h \le h$

By (2) this implies $d_{n-h} \ge n-h$

$$d_1 \le d_2 \le \ldots \underset{\underset{h}{\vee\!|}}{d_h} \le d_{h+1} \le \ldots \underbrace{\overset{\overset{n-h}{\wedge\!|}}{d_{n-h}} \le d_{n-h+1} \le \ldots \le d_n}_{h+1 \text{ vertices}}$$

So there are at least $h+1$ vertices with degree $\ge n-h$

Since $\deg(x) = h$ one of those vertices is not adjacent to x

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Denote this vertex by z.

We have $\underbrace{deg(x)}_{h} + \underset{\substack{\text{IV} \\ n-h}}{deg(z)} \geq h + n-h = n$ and $\{x,z\} \notin E(G)$

But then $deg(x) + deg(z) \geq n \overset{(3)}{>} deg(x) + deg(y)$

and this contradicts the choice of $\{x,y\} \notin E(G)$ to maximize $deg(x) + deg(y)$.

$\Leftarrow$) Assume $0 < a_1 \leq a_2 \leq \dots \leq a_n < n$ is a Hamilt. seq. with $n \geq 3$

Show that $a_i \leq i \Rightarrow a_{n-i} \geq n-i \quad \forall \quad i < \frac{n}{2}$. (*)

Via contradiction; i.e. assume the sequence does not fulfill (*). Let $h < \frac{n}{2}$ with $a_h \leq h$ and $a_{n-h} < n-h$ and construct a graph G which is not Hamiltonian and has a degree sequence $d_1, d_2, \dots d_n$ with $d_i \geq a_i$, (**) $\forall i \in \overline{1,n}$.

More precisely observe that the sequence

$$d := (\underbrace{h, h, \dots, h}_{h \text{ times}},\ \underbrace{n-h+1, \dots, n-h+1}_{(n-2h) \text{ times}},\ \underbrace{n-1, \dots, n-1}_{h \text{ times}})$$

d_h $\quad$ $d_{n-h} = n-h+1 > n-h > a_{n-h}$

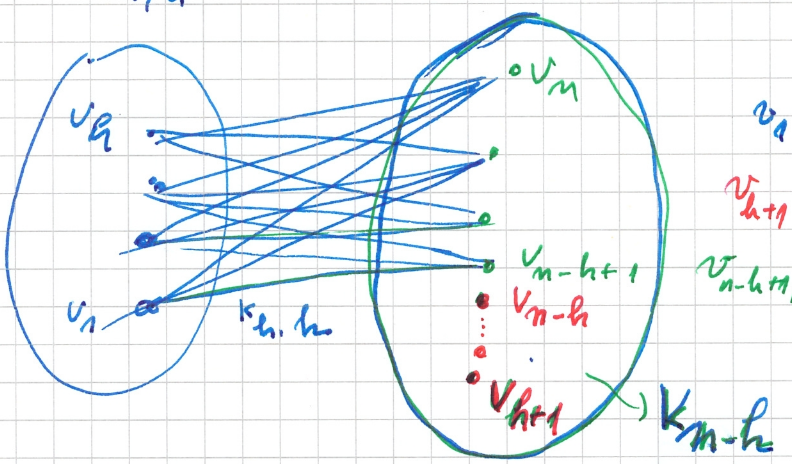
fulfills (**),

and construct a graph with degree sequence d which is not Hamiltonian

Let $V(G) = \{v_1, \dots, v_n\}$ and $\longrightarrow$

$E(G) = \{ \{v_i, v_j\} : i, j > h \} \cup \{ \{v_i v_j\} : i \leq h, j > n-h \}$

So G is the union of a K_{n-h} with vertex set $\{v_{h+1}, \dots v_n\}$ and a $K_{h,h}$ with partition $\{v_1, \dots v_h\} \cup \{v_{n-h+1}, \dots, v_n\}$.

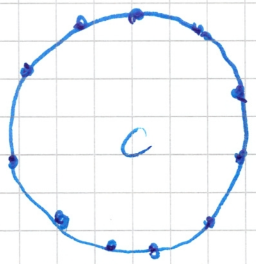


$v_1, \dots, v_h$ blue vertices
$v_{h+1}, \dots, v_{n-h}$ red vertices
$v_{n-h+1}, \dots, v_n$ green vertices

This graph is not Hamilt. because every cycle containing $v_1, \dots, v_h$ misses v_{h+1}

Indeed let C be a (Hamilt) cycle containing $v_1, \dots, v_h$

Fix an order in C (e.g. clockwise) and consider the blue vertices



The successor any blue vertex is a green vertex. But also the predecessor of a blue vertex must be a green vertex

And there exactly h blue vertices and h green vertices. So the cycle cannot contain any other vertices but blue and green. So it misses v_{h+1} and any other red vertex, and this completes the proof. $\boxed{}$