Proof of Proposition 1(a)

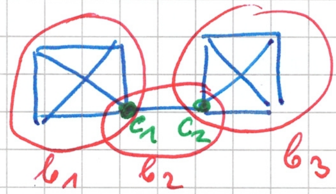
$\Rightarrow$) G is planar $\Rightarrow$ all blocks are planar (trivial)

$\Leftarrow$) Assume all block are planar
Show that G is also planar

Assume w.l.o.g. the the graph G is connected, otherwise prove the statement for every conn. comp. and draw each of them separately and "put those drawings next to each other".

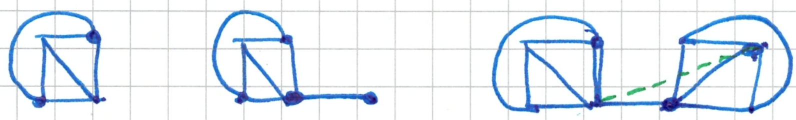
Consider bc(G). We know that bc(G) is a tree. Pick up a vertex of the tree which is a block and embed the block in the plane.
Then search the tree bc(G) starting at that block e.g. by DFS and embed the block encountered next such that the cut-vertex along with the block is being entered lies on the border of the outerface of the embedding constructed so far.

G



bc(G)

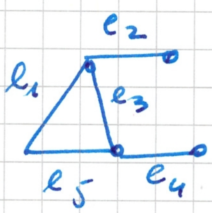
Embedding:

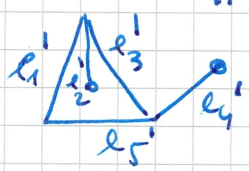


(1) (2) (3)

Examples of non-equivalent embeddings:

1





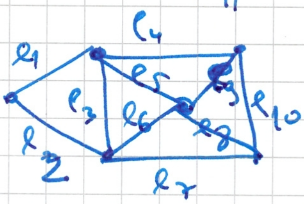
$R = (e_1\ e_5\ e_4\ e_3\ e_2)$ border of the outer face

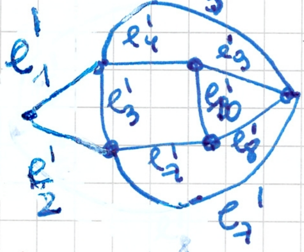
$\varphi = id$

$\varphi' : e_i \mapsto e_i' \ \forall\, i \in \overline{1,5}$ isomorphism

But $\varphi'(R) = (e_1'\ e_5'\ e_4'\ e_3'\ \underline{e_2'})$ is not the border of the outer face of H'

2





$R = (e_1\ e_2\ e_7\ e_{10}\ e_4)$

$\varphi = id$

$\varphi'(R) = (e_1'\ e_2'\ e_7',\ e_{10}',\ e_4')$

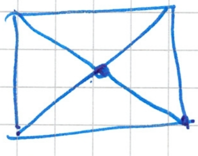
$\varphi' : e \mapsto e' \ \forall\, i \in \overline{1,10}$

and outer face of H' is

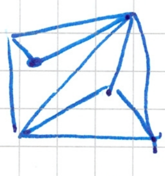
$(e_1'\ e_2'\ e_7'\ e_5') \neq \varphi'(R)$

Example of triangulations and non-triangulations

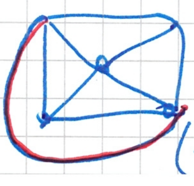
(a)



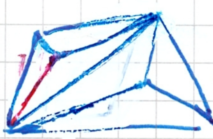
(b)



(c)



(b')



(c')

Proof of Proposition 4

Let G be a planar graph the following statements are equivalent

(i) G is maximal planar

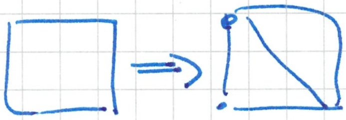
(ii) G is a plane triangulation

(iii) $|E| = 3|V| - 6$ [where $(G = V, E)$]

We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii)

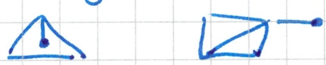
If G is max planar than the border of every edge is a triangle, otherwise we could add an edge ↯ (contradicting maximality)



(ii) $\Rightarrow$ (iii)

Count the edge-face incidencies in 2 ways:

(a) $$\sum_{e \in E} \sum_{\substack{F \text{ is face}: \\ e \in \underbrace{\delta(F)}_{\text{border of } F}}} 1 = 2|E|$$ because every edge is on the border of exactly 2 faces (otherwise the border of the face would not be a triangle



)

(b) $$\sum_{F \text{ is a face}} \sum_{\substack{e \in E: \\ e \in \delta(F)}} 1 = 3|F|$$

Thus $2|E| = 3|F| \Rightarrow F = \frac{2|E|}{3}$

Plug this in Euler's formula

$$2 = |V| - |E| + |F| = |V| - |E| + \frac{2|E|}{3} \Rightarrow |E| = 3|V| - 6$$

(iii) ⇒ (i)

Assume $|E| = 3|V| - 6$. Then G is maximal planar because otherwise we could add an edge and the graph would still be planar but violate

$E \leq 3|V| - 6$. ↯

$\boxed{\Xi}$