Proof of Corollary 12 (of Vizing and Fournier)

Let G be a graph and $S := \{v \in V(G) : \deg(v) = \Delta(G)\}$

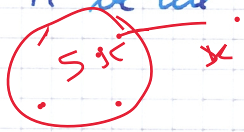
If $G[S]$ is cycle-free, then G is a Class I graph.
An edge coloring with $\Delta = \Delta(G)$ colors can be constructed in $O(|E(G)||V(G)|)$ time.

Proof Notice that in the proof of Vizing's theorem we worked with a set of colors $\{1, 2, \dots, \Delta(G)+1\}$ in order to guarantee that in every vertex x there is a missing color.
The assumption of this corollary guarantees the existence of such a missing color also if we work with colors $\{1, 2, \dots, \Delta(G)\}$.

Assume w.l.o.g. that $\Delta(G) \neq 0$. Let F be the set of edges in $G[S]$, $F := E(G[S])$. We apply induction on $|F|$.

Induction basis: $F = \emptyset$

In this case F is a stable set in G.
$\forall x \in S$ choose exactly one edge $\{x, x'\} \in E(G)$ arbitrarily. Let M be the set of all those edges, i.e.

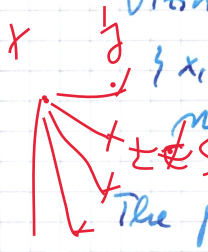


$$M = \{\{x, x'\} : x \in S\}.$$ Clearly $|M| = |S|$

According to Vizing's theorem the edges of $G-M$ can be colored with $\Delta(G-M)+1 = \Delta(G) - 1 + 1 = \Delta(G)$ colors.
Now insert iteratively $\{x, x'\} \in M$ and color those like in the proof of Vizing's theorem. Since there are no neighboring vertices of degree $\Delta(G)$ a color from $\{1, 2, \dots, \Delta(G)\}$ will be missing in x before $\{x, x'\}$ has been colored. Moreover $\deg(x') < \Delta(G)$ holds, because $x' \notin S$ since S is a stable set and $x \in S$, $\{x', x\} \in E(G)$.
So the proof like in Vizing's theorem works through.

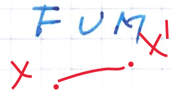
Induction step: $|F| \geq 1$

Let $x \in S'$ be a leaf in the forest $G[S]$ and let $\{x,y\}$ be the edge incident with x in $G[S]$. By the induction assumption $G-\{x,y\}$ can be colored with $\Delta(G)$ colors. We extend this coloring to also color $\{x,y\}$ with a color from $\overline{1,\Delta(G)}$ as in the proof of Vizing's theorem. The reason is that prior to coloring $\{x,y\}$ a color is missing in x and also in all other neighbors z of x but y since $z \in V \setminus S$ implies $deg(z) < \Delta$.



The proof also suggests an algorithm to construct a $\Delta(G)$-edge coloring of a graph fulfilling the assumptions of the corollary.

First construct the set S'. By a breadth first search over the components of $G[S]$ one can obtain $F \cup M$ (and also check whether $G[S]$ is a forest).



All these operation can be performed in $O(m+n)$ time.

Then a $\Delta(G)$-edge coloring of $G-(F \cup M)$ can be constructed as in the algorithmic proof of Vizing's theorem. This is done in

$O(|E(F \cup M)| \cdot |V(G)|)$ time.

Then the edges of $F \cup M$ are colored iteratively as in the proof above, one edge at a time, in the reverse order in which they were found and added to $F \cup M$, i.e. first the edges from M and then the edges leading to leaves of $G[S]$ and so on.

This can be done again in $O(|V(G)|)$ per edge so in $O(|F \cup M|\,|V(G)|) = O(|E(G)| \cdot |V(G)|)$ for all edges, leading to an overall time complexity of $O(|E(G)|\,|V(G)|)$. ∎