(Chapter 4) Planar graphs: minors

Definition 6.

A graph G contains a graph H as a **minor** *iff G contains a subgraph H' which can be contracted to H, i.e. it is possible to obtain H from H' by successive edge contractions.*
Notation: $G \succeq H$.

Remarks:

(1) The minor relationship $\succeq$ is a partial order on the (isomorphism classes) of (finite, undirected) graphs:

- (i) **reflexive:** every graph is a minor of itself,
- (ii) **transitive**: $G_1 \succeq G_2$ and $G_2 \succeq G_3$ imply $G_1 \succeq G_3$, for any three graphs G_i, $i \in \{1, 2, 3\}$,
- (iii) **anti-symmetric**: $G_1 \succeq G_2$ and $G_2 \succeq G_1$ imply $G_1 \simeq G_2$).

(2) If G contains a subdivision of H, then G contains the graph H also as a minor (if H' is a subdivision of H than H is a contraction of H'). The converse does not hold in general!

(Chapter 4) Planar graphs: the theorem of Wagner

Definition 7.

Let G be a graph and $A, B \subset V(G)$ be two disjoint sets of vertices in G, i.e. $A \cap B = \emptyset$. The cut defined by A and B in G is denoted by $\delta(A, B)$ and is defined as
$\delta(A, B) := \big\{\{u, v\} \in E(G) \colon u \in A, v \in B \text{ or } u \in B, v \in A\big\}$.

Theorem 7.

Let G and H be two connected graphs. $G \succeq H$ holds iff there exists a mapping $\phi \colon V(G) \to V(H)$ such that for the pre-images $\phi^{-1}(v)$ of $v \in V(H)$ the following statements hold

(i) *$G[\phi^{-1}(v)]$ is connected for all $v \in V(H)$.*

(ii) *$\forall \{u, v\} \in E(H)$ the cut $\delta\big(\phi^{-1}(u), \phi^{-1}(v)\big)$ in G is non-empty.*

If $\Delta(H) \leq 3$ holds for a graph H, then some graph G contains H as a minor iff G contains H as a subdivision.

Corollary 8.

(Theorem of Wagner, 1937)
A graph G is planar iff it contains neither K_5 nor $K_{3,3}$ as a minor.

(Chapter 4) Planar graphs: Wagner's conjecture and the graph minor theorem

Remark: For every fixed graph H, it can be tested in polynomial time whether H is a minor of an input graph G.
N. Robertson and P.D. Seymour, Graph Minors. XIII. The disjoint paths problem, *Journal of Combinatorial Theory (B)* **63 (1)**, 1995, 65-110.

Wagner's theorem characterizes the planar graphs in terms of forbidden minors.

Wagner's conjecture: Every family of graphs that is closed under minors can be defined by a finite set of forbidden minors.

The Robertson–Seymour theorem (RSTh) or graph minors theorem: Wagner's conjecture was proved by Robertson and Seymour in a series of twenty papers (1983–2004, more than 500 pages).

Equivalent formulations of RSTh:

(i) The graph minor relationship is a well-quasi-ordering in the set of finite undirected graphs, i.e. there exists neither an infinite descending chain nor an infinite antichain in this partial order.

(ii) In any infinite set S of graphs, there must be a pair of graphs one of which is a minor of the other. (Lovasz 2005)

(Chapter 4) Planar graphs: Duality

Definition 8.

Consider a planar embedding of a planar graph $G = (V, E)$ with set of faces $\mathcal{R}$. The **geometric dual** *G^* of G is a multigraph with vertex set V^* obtained as a collection of points in the interior the faces in $\mathcal{R}$ with exctly one point per face. The edge set E^* of G^* is a collection of edges e^* associated to edges of G as follows. $\forall e \in E(G)$ consider the faces $f, f' \in \mathcal{R}$ which contain e in their borders and join the vertices of V^* selected in the interiors of f, f', respectively, by a Jordan curve e^*, such that*

(i) *e^* intersect e in an inner point,*

(ii) *e^* is contained in the union of f and f'.*

If e is a bridge, then let $e*$ be a loop.

A planar graph is called **self-dual** *if $G \simeq G^*$ for some geometric dual related to a planar embedding of G-.*

Remarks:

(1) The multigraph G^* contains (a) loops if G contains bridges and (b) multiple edges if there are pairs of regions in $\mathcal{R}$ sharing more than one edge in their respective bordes.

(2) There is a bijective mapping between E and E^*.

(3) G^* depends on the planar embedding of G. Equivalent embeddings lead to isomorphic geometric duals, but there might be non-equivalent embedding leading to isomorphic geometric duals.

(4) For any planar embedding of a planar graph G, G^* is connected.

(5) If G is planar and connected, then $(G^*)^* \simeq G$ holds.

Proposition 9.

The geometric dual G^* corresponding to a planar embedding with at least 3 faces of a 2-connected planar graph G, is 2-connected.

The converse is in general not true.

(Chapter 4) Planar graphs: Duality and connectivity

Proposition 10.

The geometric dual G^* corresponding to a planar embedding of a 3-connected planar graph G, is a simple graph and it is 3-connected.

Definition 9.

A graph is called polyhedral iff it is isomorphic to a 3-dimensional bounded convex plohedron (polytope), where the vertices and edges of the graphs are bijectively mapped to the vertices and the edges of the polyhedron, respectively, and the regions of the graph are mapped to the surfaces of the polyhedron.

Theorem 11.

(Steinitz 1922)
A graph is polyhedral iff it is planar and 3-connected.

(Chapter 4) Planar graphs and platonic solids

Definition 10.

A polyhedron is called regular or a **platonic solid** *iff it has regular congruent polygonal faces with the same number of faces meeting at each vertex. Thus the graphs corresponding to platonic solids are regular.*

Theorem 12.

There are exactly 5 *platonic solids: the tetrahedron, the octahedron, the hexahedron, the dodecahedron and the icosahedron.*

Corollary 13.

The polyhedral graphs associated to the platonic solids are exactly those polahedral graphs which are regular together with the corresponding geometric duals.

The (graph associated to the) tetrahedron is self-dual, the (graphs associated to the) octahedron and the hexahedron are the geometric duals of each other, and the (graphs associated to the) icosahedron and the dodecahedron are the geometric duals of each other.

(Chapter 4) Planar graphs: the combinatorial dual

Theorem 14.

Let G be a planar graph and G^* a geometric dual of G related to a planar embedding of G. Then $C \subseteq E(G)$ is a cycle in G iff the corresponding set of edges C^* is an inclusion minimal separating set of edges in G^*.

Definition 11.

Let $G = (V, E)$ be a graph. $G^* = (V^*, E^*)$ is called a combinatorial dual of G iff there exists a bijection $\phi\colon E \to E^*$ such that $C \subseteq E$ is a cylce in G if and only if $C^* := \phi(C) \subseteq E^*$ is a minimum separating set of edges in G^*.

Corollary 15.

Every geometric dual of a graph G is also a combinatorial dual of it.

Theorem 16.

(Whitney 1933)
A graph G has a combinatorial dual iff G is planar.

(Chapter 4) Planar graphs: the recognition problem

Input: A graph $G = (V, E)$
Question: Is G planar?

This problem is solvable in linear time.

Embedding algorithms construct a planar embedding of G if G is planar or show the existence of a forbidden subgraph (path addition, vertex addition, edge addition or construction sequence methods).

First linear time algorithm (path addition):
J. Hopcropft and R.E. Tarjan, Efficient planarity testing, *J. of the ACM* **21(4)**, 1974, 549–568.

State of the art (edge addition):
J.M. Boyer and W.J. Myrwald, On the cutting edge: simplified $O(n)$ planarity by edge addition, *J. of Graph Algorithms and Applications* **8(3)**, 2004, 241–273.

Another state of the art algorithm:
H. de Fraysseix, P. Ossona de Mendez, P. Rosenstiehl, Trémaux Trees and Planarity, *Intern. J. of Foundations of Computer Science* **17(5)**, 2006, 1017-1030.