(Chapter 4) Planar graphs: definitions and elementary concepts

Definition 1.

A graph G is called **planar** *if there exists a* **planar embedding** *of G in the plane such that*

(i) *the vertices of G are points in the plane*

(ii) *the edges of G are Jordan arcs in $\mathbb{R}^2$ (or open Jordan curves) such that every Jordan arc intersects $V(G)$ only in its endpoints and any two different Jordan arcs intersect in at most one endpoint.*

A **plane graph** *is a planar embedding of some planar graph.*

Recall: A **Jordan arc** is the image of a homeomorphic mapping of $[0, 1]$ in some topological space (e.g. $\mathbb{R}^2$). A **mapping** $f : A \to B$ is called **homeomorphic** iff $f : A \to f(A)$ is a continuous bijection and f^{-1} is also continuous (where A, B are subsets of topological spaces).

Remark: Since the plane $\mathbb{R}^2$ and the sphere from which the north pole has been removed are homeomorphic (stereographic projection), the planarity of a graph is equivalent to its embeddability on the surface of a sphere without the north pole such that condition (i) and (ii) above are fulfilled.

Definition 2.

If the edges are removed from a planar embedding of a planar graph, then the plane would decompose in simple connected areas called **faces** *(or regions) of which exactly one is unboundend. The unbounded face is called* **the outer face**. **The border of a face** *consists of all edges which belong to the closure of this face.*

Remark: Consider an arbitrary face F of a planar embedding ϕ of a planar graph G. There always exists a planar embedding ϕ' of G in which F is the outer face, i.e. in which the borders of the outer face F' of ϕ' and the face F in ϕ are the same. (Show this by means of the stereographic projection!)

Proposition 1.

(a) *A graph is planar iff all its blocks are planar.*

(b) *A planar graph is 2-connected iff the border of every face is a cycle.*

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Definition 3.

Let ϕ and ϕ' be two **embeddings** *of a planar graph G in the plane, i.e. ϕ and ϕ' are isomorphisms of G and two plane graphs H and H', respectively. ϕ and ϕ' are called* **equivalent** *iff $\phi' \circ \phi^{-1}$ is an isomorphism of H and H' such that for all sequences of edges R in E the following holds: $\phi(R)$ is a border of a region in H iff $\phi'(R)$ is a border of a region in H' in the same order or in the reversed order.*

Proposition 2.

(Wagner 1936, Fáry 1948)
For every embedding of a planar graph G in the plane there exists an equivalent planar embedding in which all edges are straight lines, i.e. the Jordan arcs in the definition of the planar embedding are linear mappings.

(See W.T. Tutte, How to draw a graph, *Proceedings of the London Mathematical Society* **13**, 1963, 743–767 for a proof.)

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Remark: Any two embeddings of a path P_n, a cycle C_n and of $K_{1,3}$ are equivalent, respectively. In such a case we say that the corresponding graphs are **uniquelly embeddable**.

Theorem 3.

(Whitney 1932)
A 3-connected graph is uniquely embeddable.

(See R. Diestel, *Graph Theory*, fourth edition, Springer, 2012, pp. 100 for a proof.)

Recall: (Euler's formula) For a planar, connected graph $G = (V, E)$ the equality $|V| - |E| + |F| = 2$ holds, where F is the set of faces of a planar embedding of G. If G is planar with $|G| \geq 3$, then $|E| \leq 3|V| - 6$ holds. If G is planar, triangle free with $|G| \geq 3$, then $|E| \leq 2|V| - 4$ holds.

Observation: K_5 has 10 edges, i.e. more than $3|V(K_5)| - 6 = 9$. Thus K_5 is not planar!

$K_{3,3}$ has 9 edges, i.e. is more than $2|V(K_{3,3})| - 4 = 8$. Thus $K_{3,3}$ is not planar!

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Definition 4.

A planar graph $G = (V, E)$ is called **maximal planar** *if the graph $G + \{x, y\}$ is not planar $\forall x, y \in V$, $\{x, y\} \notin E$. A* **triangulation** *of the plane (or a triangle graph) is a plane graph with the following property: the border of every face (including the outer face) is a triangle, i.e. a cycle of length 3.*

Proposition 4.

For a planar graph $G = (V, E)$ the following statements are equivalent:

(i) *G is maximal planar,*

(ii) *G is a plane triangulation ,*

(iii) $|E| = 3|V| - 6$.

(Chapter 4) Planar graphs: the theorem of Kuratowski

Definition 5.

Let a graph G be obtained from a graph H through iterative edge divisions, i.e. iterative insertions of a new vertex of degree two in some edge of the graph. Equivalently, G results from the substitution of edges of H by paths of length larger than 1, where paths corresponding to different edges may just share some endpoint. Then G is called a **subdivision** *(or a* **topological subgraph***) of H. Two graphs are called* **homeomorphic** *if they are both subdivisions of a third graph.*
A graph H **contains** *G* **as a subdivision** *(or as a topological subgraph) if H contains a subgraph G' which is a subdivision of G.*

Observation: G is planar $\Longleftrightarrow$ every subdivison of G is planar.

Thus G is planar $\Longrightarrow$ G contains no subdivision of K_5 or $K_{3,3}$.

Theorem 5.

(Kuratowski 1930)
A grapf is planar iff it contains no subdivisons of K_5 and $K_{3,3}$.

The proof is due to Thomassen 1981 and is based in 2 lemmas.

(Chapter 4) Planar graphs: the theorem of Kuratowski

A lemma we have already seen in Chapter 3:

Every 3-connected graph G with at least 5 vertices contains an edge e such that G/e is 3-connected.

Lemma 6.

Consider some graph G and an edge $e \in E(G)$. If G/e contains a subdivision of K_5 or $K_{3,3}$, then G also contains such a subdivision, respectively.