Minor (1)

$H \preceq G \iff G \supseteq H'$ and H' can be contracted to H

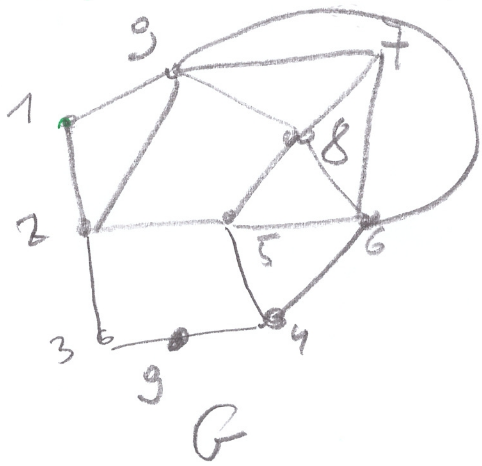
(2) Subdivision

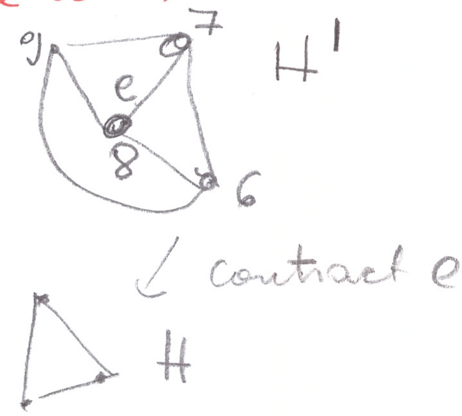
H is contained in G as a subdivision $\iff G \supseteq H'$, H' is a subdivision of H

H is contained in G as a subdivision $\implies$ H is contained in G as a minor (but not vice-versa)

Examples

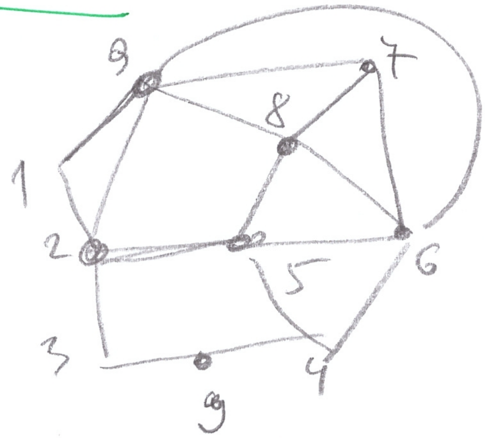
Minor

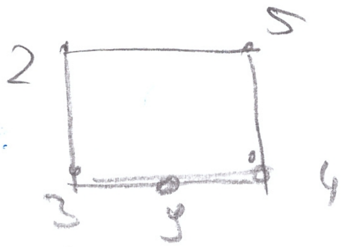




H is a minor of G

Subdivision





H' is a subdivision of

~~On~~ planar graphs, Wagner's conjecture and graph minor theorem

Equivalent definition of a minor:

A minor of an undirected graph G is any graph that may be obtained from G by a sequence of zero or more contractions of edges of G and deletions of edges and vertices of G.

Definition of a well-quasi-ordering

A preorder (a relation which reflexive and transitive) is said to form a well-quasi-ordering iff it contains neither an infinite descending-chain nor an infinite antichain.

Example: the usual ordering on the non-negative integers $\mathbb{Z}_+$ but not over $\mathbb{Z}$

Forbidden minor characterisations

A family F of graphs is said to be closed under the operation of taking minors if every minor of a graph in F also belongs to F

If F is a minor closed family, then let $\mathcal{S}$ be the set of graphs that are not in F, $\mathcal{S} = F^c$. According to RSTh there exists a finite set H of ~~minimal~~ minimal elements in $\mathcal{S}$. These elements form a forbidden graph characterization of F: The graphs in F are exactly the graphs that do not have a graph in H as a minor.

Example:

Let $\mathcal{F}$ be the family of planar graphs.

It is closed under minors because every graph obtained ~~by~~ from a planar graph by removing vertices and/or edges and edge contractions is planar.

Thus planar graphs have a forbidden minor characterization; according to Wagner's theorem the ~~set~~ H of minor-minimal nonplanar graphs contains exactly 2 elements K_5 and $K_3,$

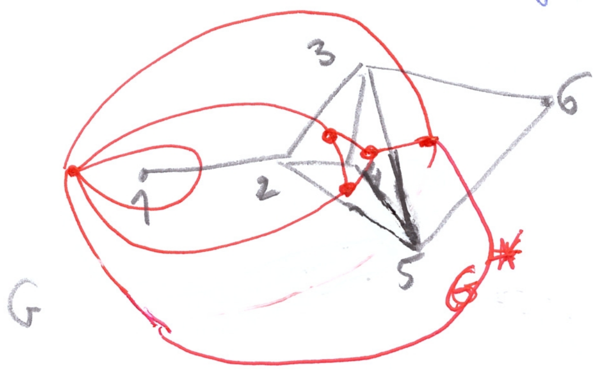
Duality in planar graphs

Example

(1)

In general G^* is not a simple graph, it contains loops and multi-edges.

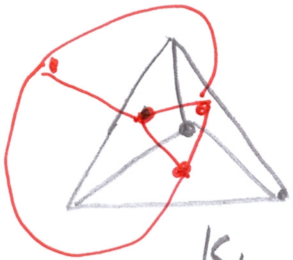
Observe $\delta(G^*)=3$



(2)

$K_4 \simeq K_4^*$

thus K_4 is self-dual

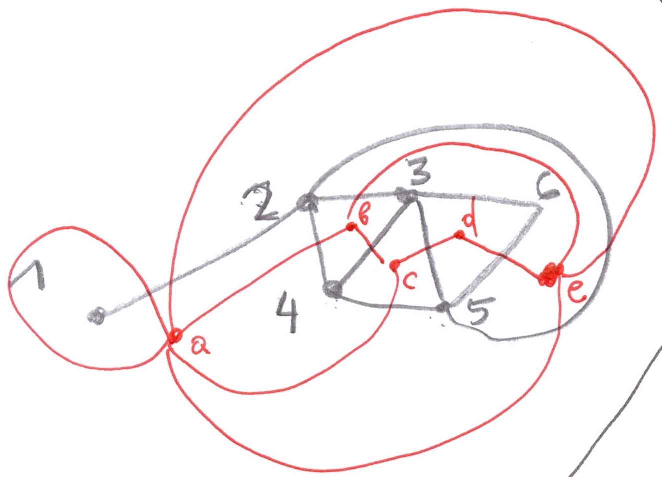


K_4 K_4^*

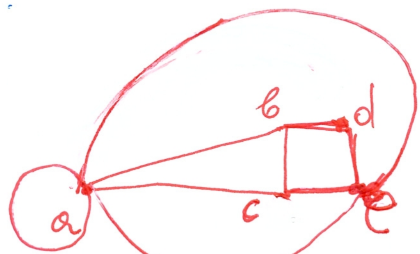
(3) ~~Are~~ other embedding for G of Example 1 and the corresponding dual $\bar{G}^*$

Observe that $\bar{\bar{G}}^* \not\simeq G^*$

and $\bar{G}^* \simeq \bar{\bar{G}}^*$



$\bar{G}^*$



$\bar{\bar{G}}^*$

$\deg_{\bar{G}^*}(d)=2$ but there is no vertex of degree 2 in G^*

Proof of Proposition 3

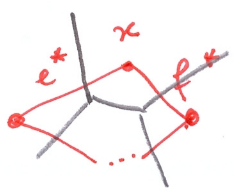
Let G be 2-connected and planar.
Consider a planar embedding of G with 3 regions (assume such an embedding exists)

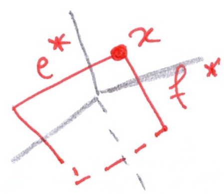
Case I: The planar embedding as above has exactly 3 faces.
Then any 2 of them have at least an edge in common. $\Rightarrow$ The corresponding dual is a K_3 with (possibly) multiple edges, and K_3 is 2-connected.

Case II Assume that the considered planar embedding has more than 3 regions.
Let G^* be the corresp. geometric dual.

We show $\forall\, e^*, f^* \in E(G^*)$ which are not parallel (i.e. do not have common end points) belong to a common cycle (saw this in Exercises, Exercise 6)

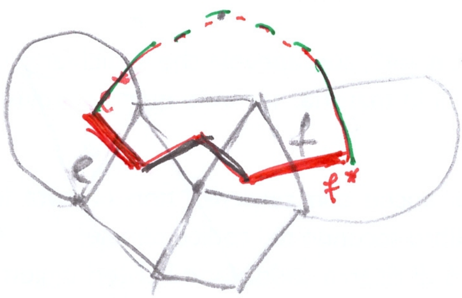
Case IIa) $e^* \cap f^* = \{x\}$ where x belongs to some region R of G. Then follow the neighboring regions of x in clockwise or counter-clockwise order



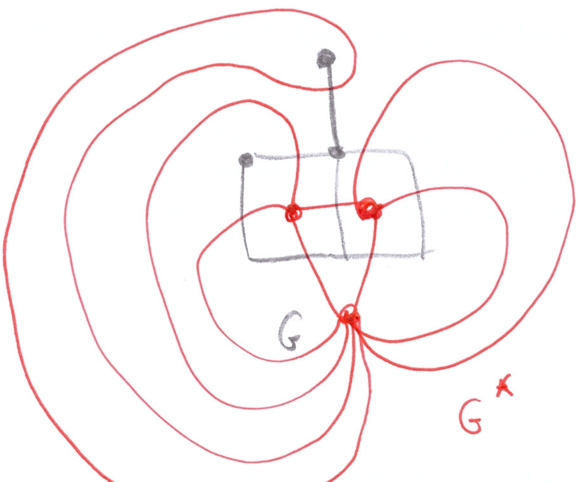


Case II b $\quad e^* \cap f^* = \emptyset$

The corresponding edges e and f lie on a cycle which divides the plane in an inner and an outer area.
Both in the inner area and in the outer area there is a chain of regions of G which "connect" e^* and f^*. These 2 chains form the required cycle



The converse statement is not ~~true~~ true, i.e.
if the geometric dual of graph is 2-connected, then the original planar graph is not necessarily 2 connected with at least 3 regions.



G^* is 2 connected because it contains k_3 as a subgraph

G is however not 2 connected; it has a vertex of degree 1.

Proof of Corollary 13 (Sketch)

$\Rightarrow$) Ass.) Let G be a polyhedral graph associated to a platonic solid S.

Show: G and G^* are regular

G is regular because in every vertex of S meet the same number of faces ~~of~~ S
$\rightarrow$ all vertices of G have the same degree

G^* is regular because the faces of S are congruent $\Rightarrow$ have the same number of edges on their border $\Rightarrow$ all vertices of G^* have the same degree

$\Leftarrow$) Assume G is polyhedral and G, G^* are regular
Show: G is associated to a platonic solid

G^* is regular $\Rightarrow$ (i) all faces of S have the same nr of edges in their border

G is regular $\Rightarrow$ (ii) in every vertex of S' meet the same number of faces.

But (i) and (ii) are not enough.
It can be shown that a polyhedron S with property (i) and (ii) can be transformed by a homeomorphic transf. to a Platonic solid.

Platonic solids: tetrahedron, Cube, Octahedron, dodecahedron, icosahedron

dual dual dual