Chapter 6: Perfect graphs

We have seen (or will see in the exercises!) that the gap between $\chi(G)$ and $\omega(G)$ can be arbitrarily large. ($\chi(G) \geq \omega(G)$ holds trivially for every graph). [For example the Mycielski graphs M_k fulfill $\chi(M_k) = k$, are triangle free ($\Rightarrow \omega(M_k) = 2$), and have $3 \cdot 2^{k-2} - 1$ vertices.] Exercises

On the other hand there are of course graphs for which $\chi(G) = \omega(G)$ holds; given a graph with $\chi(G) > \omega(G)$, just add to it a disjoint clique of cardinality $\chi(G)$ to obtain a graph G' for which $\chi(G') = \omega(G')$ holds.

There is however a class of graphs for which $\chi(G) = \omega(G)$ holds, and even more, the equality holds for every induced subgraph of G.

Def (Shannon 1956, Berge 1961)

A graph G is called *perfect* iff $\chi(H) = \omega(H)$ holds for every induced subgraph $H \trianglelefteq G$.

Examples Bipartite graphs (and hence also trees) are perfect. An induced subgraph of a bipartite graph is again a bipartite graph and for bipartite graphs we have

$$\omega(G) = \chi(G) = \begin{cases} 2 & \text{if } E(G) \neq \emptyset \\ 1 & \text{if } E(G) = \emptyset \end{cases}$$

Proposition 6.1 Complements of bipartite graphs are perfect graphs.

Observation: An induced subgraph of a complement of a bipartite graph is again a complement of a bipartite graph.

We say that the property "being complement of a

Bipartite graph is hereditary.

Indeed let G be a bipartite graph and consider an induced subgraph of G^c G with vertex set V_1.

Then $(G[V_1])^c = G^c[V_1]$ because $\forall x,y \in V_1$
$\{x,y\} \in E(G) \iff \{x,y\} \notin E(G^c)$.

With the above observation it is enough to show
$\chi(G^c) = \omega(G^c)$ for some bipartite graph G.

[This is of course equivalent to $\theta(G) = \alpha(G)$, where $\theta(G)$ is the clique partition number of G, i.e. the smallest nr. of cliques in which the vertices of G can be partitioned], i.e.

$$\theta(G) = \min\Big\{k : V(G) = \dot{\bigcup_{1\le i\le k}} V_i \text{ s.th. } G[V_i] \text{ is a clique } \forall i \in \overline{1,k}\Big\}$$

Lemma: For a triangle-free graph G without isolated vertices the equality $\rho(G) = \theta(G)$ holds.

~~Assume we have proved this lemma~~
Assume the bip graph has no isolated vertices

(a) The equality $\theta(G) = \alpha(G)$ follows then from König (1) $\tau(G) = \nu(G)$ in combination with Gallai

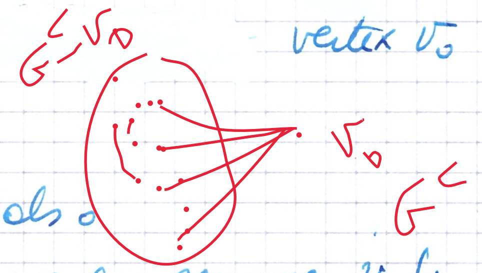
Let $|G| =: n$

(2) $\tau(G) + \alpha(G) = n$ and (3) $\rho(G) + \nu(G) = n$

[(1)+(2) $\Rightarrow$ (4) $\nu(G) + \alpha(G) = n$; then (4) and (3) imply $\rho(G) = \alpha(G)$]

because then $\alpha(G) = \rho(G)$ and with $\rho(G) = \theta(G)$ (Lemma) we get $\alpha(G) = \theta(G)$.

(b) If the bip. graph G has an isolated vertex v_0



then $\deg_{G^c}(v_0) = |V(G)| - 1$

Hence $\chi(G^c) = \chi(G^c - v_0) + 1$ and also
$\omega(G^c) = \omega(G^c - v_0) + 1$. So we can remove v_0 from G^c, prove the statement for the remaining graph, and

and repeat this step as long as G has isolated vertices. $\boxed{\square}$

Proof of the lemma: (1) $\rho(G) \le \theta(G)$

~~Let~~ G be triangle-free without isolated vertices and let $\dot{\bigcup}_{i=1}^{\theta(G)} V_i = V(G)$ be an optimal partition of V into cliques, i.e. $G[V_i]$ is a clique, $\forall i \in \overline{1, \theta(G)}$.

Since G triangle free, we have $|V_i| \le 2$ $\forall i \in \overline{1, \theta(G)}$

Construct an edge cover by taking (a) the edge in $G[V_i]$ $\forall i$ for which $|V_i| \ge 2$ and (b) an arbitrary edge incident to the single vertex of V_i, $\forall i \in \overline{1, \theta(G)}$ with $|V_i| = 1$. The card. of this edge cover is $= \theta(G)$

$$\Longrightarrow \rho(G) \le \theta(G).$$

(2) $\rho(G) \ge \theta(G)$

Let E_1 be an edge cover with $|E_1| = \rho(G)$. (an optimal EC).

Take a maximal matching M in (E_1), $M = \{\{x_1, y_1\}, \ldots \{x_k, y_k\}\}$ for some $k \le \rho(G)$.

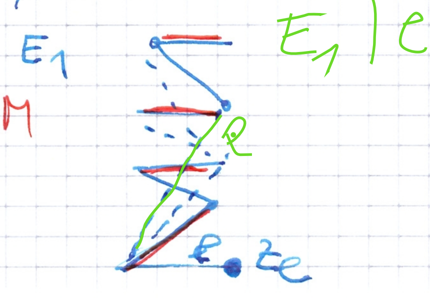
$\forall$ $e \in E_1 \setminus M$ let z_e be the end vertex of e which is not in $\bigcup_{i=1}^{\rho(G)} \{x_i, y_i\}$ (there should be such a vertex otherwise $E_1 \setminus \{e\}$ would an even smaller edge cover than E_1 $\lightning$).

~~The~~ Set $V_i = \{x_i, y_i\} \subseteq V$ $\forall$ $1 \le i \le k$

and $V_e := \{z_e\}$ $\forall$ $e \in E_1 \setminus M$ for which such a z_e is defined.

Then $\dot{\bigcup}_{1 \le i \le k} V_i \;\dot{\cup}\; \dot{\bigcup}_{e \in E_1 \setminus M} V_e$ is a partition of V into cliques with exactly $\rho(G)$ cliques in it.

$$\Longrightarrow \theta(G) \le \rho(G)$$



$\boxed{\square}$

Def A Helly family of order k is a set system $(E, \mathcal{F})$ with $\mathcal{F}$ being a collection of subsets of E, such that $\forall$ (finite) $\mathcal{G} \subseteq \mathcal{F}$ with $\bigcap_{X \in \mathcal{G}} X = \emptyset$, there exists an $\mathcal{H} \subseteq \mathcal{G}$ such that $\bigcap_{X \in \mathcal{H}} X = \emptyset$ and $|\mathcal{H}| \leq k$.

(*) In some definition finiteness is not required, yielding a more restrictive property.

The k-Helly property is the is the property of being a Helly family of order k.

If $k=2$ we simply omit the index k and use the terms Helly family and Helly property.

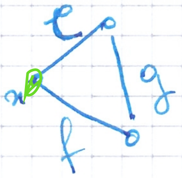
Observation Let $G=(V,E)$ a triangle-free graph. Then the set of its edges has the Helly property, i.e. (V, E) is a Helly family.

Proof We use an equivalent formulation of the definition above.

(V,E) is a Helly family $\Longleftrightarrow$ $\forall$ (finite) $E_1 \subseteq E$, $e \cap f \neq \emptyset \ \forall e,f \in E_1 \Rightarrow \bigcap_{e \in E_1} e \neq \emptyset$

So we show the property on the r.h.side of the above equivalence.

Let $E_1 \subseteq E$ be such that $e \cap f \neq \emptyset \ \forall e,f \in E_1$. Assume w.l.o.g. $|E_1| \geq 2$ and let $e,f \in E_1, e \neq f$, let $x \in e \cap f$. Assume by contradiction that $\bigcap_{e \in E_1} e = \emptyset$. Then $\exists\, g \in E_1$ with $x \notin g$. On the other hand $g \cap e \neq \emptyset$ and $g \cap f \neq \emptyset$.



But then e, f, g build a triangle in G. $\lightning$

Proposition 6.2 Line graphs of bipartite graphs are perfect.

Proof First observe that the property of being a line graph of a bipartite graph is hereditary.

Let G be ~~a~~ bipartite graph and $L(G)$ it's line graph

$$V(L(G)) = E(G) \quad E(L(G)) = \{ \{e,f\} : e,f \in E(G),\ e \cap f \neq \emptyset \}$$

Consider an induced subgraph of $L(G)$, induced by some set $E_1 \subseteq E(G)$. Then $L(G)[E_1]$ has vertex set E_1 and two elements $e,f \in E_1$ are connected by an edge in $L(G)[E_1]$ iff $e \cap f \neq \emptyset$. But this graph is the line graph of the subgraph $H = (V(G), E_1)$ of G.

So $L(G)[E_1] = L(H)$, and since every subgraph of a bipartite graph is bipartite graph, $L(G)[E_1]$ is the line graph of the bipartite graph H.

Given the hereditary property, it is enough to show $\chi(L(G)) = \omega(L(G))$ for arbitrary bipartite graphs G. G is the root graph of $L(G)$

But $\chi(L(G)) = \chi'(G)$ and $\omega(L(G)) = \Delta(G)$ due to the Helly property of the edge set of G (G is a triangle-free graph, because it is bipartite.

Thus $\chi(L(G)) = \omega(L(G))$ is equivalent to $\chi'(G) = \Delta(G)$ which holds due to Th. of König for bipartite graphs and their chromatic number.

$\boxtimes$

6

Lemma 6.5 A graph G is perfect iff $\forall$ induced subgraph $H \trianglelefteq G$ there exists a stable set $S(H)$ in H such that $\forall$ maximum clique C of H $C \cap S \neq \emptyset$ holds.

Proof Notice that 1) a maximum clique is a clique of max cardinality (this is different from a maximal clique).

The statement of the lemma can be equivalently formulated as follows

2) $\forall$ induced $H \trianglelefteq G$ $\exists$ stable $S(H)$ in G with $\omega(H - S(H)) \leq \omega(H) - 1$. (*)

$\Rightarrow$) Let G be perfect and let $H \trianglelefteq G$ be some induced subgraph of G. Consider a $\chi(H)$-coloring c of vertices in H, and a color set F for this coloring, i.e. $F = \{x \in H : c(x) = i\}$ for some $i \in \overline{1, \chi(H)}$. Further let C be a maximum clique in H. Since $\chi(H) = \omega(H)$ (because G is perfect) and $\forall y, z \in C$ $c(y) \neq c(z)$ all colors in $\overline{1, \chi(G)}$ are used by the vertices in C. $\Rightarrow F \cap C \neq \emptyset$. Set $S(H) = F$ showing the original statement!

$\Leftarrow$) We assume that (*) holds and show that G is perfect, i.e. $\chi(H) = \omega(H)$, $\forall H \trianglelefteq G$ (induced subgraph)

We show this by induction on H:

Basis: If $|H| = 1$ or 0 then $\chi(H) = \omega(H) = 1$ (or $\chi(H) = \omega(H) = 0$).

Consider now some $H \trianglelefteq G$ with $|H| \geq 2$.

By (*) $\exists$ S stable set in H fulfilling (*). Since $S \cap C \neq \emptyset$ $\forall$ maximum clique in H and $\omega(H) \geq 1$ we have $S \neq \emptyset$.

So $|H - S| < |H|$ and the induction hypothesis applies for $H - S$.

Thus $\boxed{\chi(H) \underset{(**)}{\leq}} \chi(H-S)+1 \underset{\text{ind. hypothesis}}{=} \omega(H-S)+1 \underset{(***)}{\leq} \boxed{\omega(H)}$

where $(**)$ holds because for every $\chi(H-S)$ colouring of $H-S$ we can color all vertices of S by the same additional color and obtain a feasible coloring of H,

and $(***)$ holds because due to $(*)$ for every maximum clique C of H, $|C| = \omega(H)$, $C \cap S \neq \emptyset$ holds, so any maximum clique of $H-S$ is not a maximum clique of H and therefore $\omega(H-S) < \omega(H) \Longrightarrow \omega(H-S)+1 \leq \omega(H)$

Thus $\chi(H) \leq \omega(H)$. Since $\omega(H) \leq \chi(H)$ holds for every graph we get $\chi(H) = \omega(H)$.

Some Optimisation problems for perfect graphs: simple examples

- bipartite graphs

Computation of $\omega(G)$ is trivial, coloring $\chi(G)$ is also trivial, (Solved by the same algorithm as the recognition of bipartite graphs). The theorem of König states that $\tau(G) = \nu(G)$ and in the constructive proof of this theorem suggests the construction of an optimal vertex cover C starting from an optimal matching M (the latter is a poly. solvable problem, e.g. Combinatorial Optimization 1). The complement $V \setminus C$ is then an optimal stable set. The matching edges and the vertices which are not matched by M build an optimal partition into cliques (because of $\theta(G) = \rho(G)$ and Gallai $\rho(G)+\nu(G)=|G|$

- Line graph of bipartite graphs

The optimization problems become tractable as soon as the corresponding bipartite graph (root graph) of the line graph is known. This graph can be constructed in linear time and the recognition of line graphs is solveble in linear time, e.g.

P.G.H. Lehot, An optimal algorithm to detect a line graph and output its root graph, J. ACM 21, 1974, 569–575.

$\omega(L(G)) = \Delta(G)$ and $\forall v_0 \in G$ with $\deg(v_0) = \Delta(G)$ we have $G' = \{e \in E(G) : v_0 \in e\}$ is a maximum clique in $L(G)$.

A max. matching in G corresponds to a max stable set in $L(G)$. Since $\chi(L(G))$-coloring of $L(G)$ correspond to a $\Delta(G)$-edge-coloring of G and the later can be constructed in polynomial time (constructive proof of Konigs theorem)

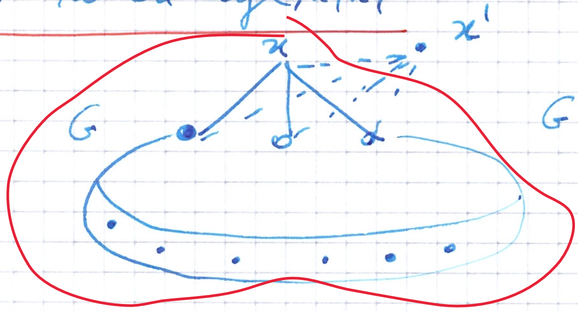
A minimum vertex cover of G (which can be constr. in poly time) determines an optimal partition into cliques in $L(G)$

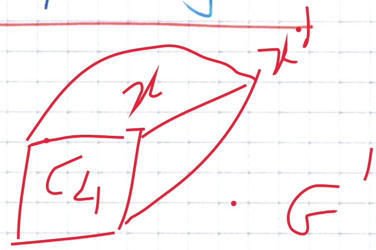
Determing $\tau(\cancel{G}\, L(G))$ is also simple. exercise

Our next goal is the proof of Theorem 8 ~~6~~ 8 (perfect graph theorem)

Def. Let G be a graph and $x \in V(G)$ a vertex and let G' be obtained from G by adding a vertex x' and joining it to x and all the neighbors of x.

We say that G' is obtained from G by expanding the vertex x to an edge $\{x, x'\}$





Lemma Every graph obtained from a perfect graph by expanding a vertex is again perfect.

Proof: Induction on the order of the considered perfect graph.

Basis: Expanding the vertex of K_1 yields K_2, which is perfect. x — x'

For the induction step let G be a non-trivial perfect graph and let G' be obtained from G by expanding a vertex x to an edge $\{x, x'\}$. It is enough to show that $\chi(G') \leq \omega(G')$ and this would imply the perfectness of G'.

Why? Every proper induced subgraph H of G' is either (a) isomorphic to an induced subgraph of G (if $V(H) \subseteq V(G)$ or $V(H) \subseteq V(G \setminus x) \cup \{x'\}$) or x'

(b) obtained from a proper induced subgraph of G by expanding x to edge $\{x, x'\}$ (if ~~$V(H) \subseteq \{x, x'\}$~~ $x \in V(H)$ and $x' \in V(H)$). In case (a) H is perfect due to the assumption that G is perfect; in case (b) H is perfect due to the induction hypothesis. So H can be colored with $\omega(H)$ colors.

$\chi(H) \leq \omega(H)$ ✓

$H_1 := H \setminus \{x, x'\}$; $H_1 \trianglelefteq G \setminus x$

$\Longrightarrow \chi(H) = \omega(H)$

Let now $\omega(G) =: \omega$; then $\omega(G') \in \{\omega, \omega+1\}$, depending whether there is a maximum clique C in G containing x (in which case $C' = C \cup \{x'\}$ is a max. clique in G') or whether there is no such a max. clique in G. x'

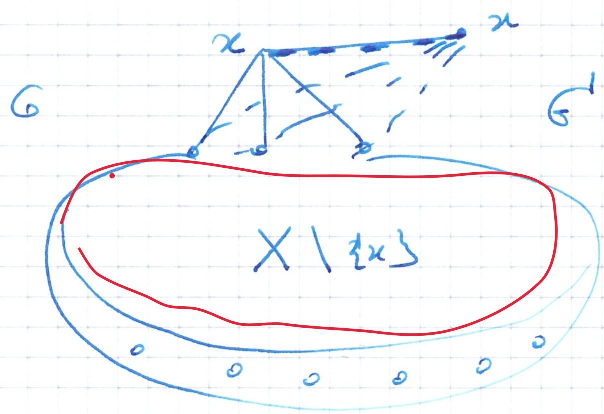
If $\omega(G') = \omega + 1$ then $\chi(G') \leq \chi(G) + 1 = \omega + 1 = \omega(G')$ and we are done. ✓

$\Longrightarrow G$ is perfect

So let us assume $\omega(G') = \omega$. Then x does not lie in any $K_\omega \subseteq G$, together with x'. This would yield ~~$\omega(G')=\omega+1$~~ $K_{\omega+1} \subseteq G'$. Color G with ω colors. Let X be the color class of x, i.e. the set of vertices in G having the same color as x.

(*) Every $K_\omega \subseteq G$ meets the color class X but does not contain x. This implies that $H := G - (X \setminus \{x\})$ has clique number $\omega(H) < \omega$.

Indeed a clique of card ω in H would contain x (because it would use all ω colors), thus contradicting statement (*).



So we can color H with $\omega - 1$ colors. Since X is independent also $(X \setminus \{x\}) \cup \{x'\}$ is independent. But $V(G' - H) = (X \setminus \{x\}) \cup \{x'\}$, so we can extend our $(\omega-1)$-coloring of H to an ω-coloring of G' showing that $\chi(G') \leq \omega = \omega(G')$ as desired. ∎

Proof of Theorem ~~6.~~8

We apply induction on $|G|$ to show that the complement $\overline{G}$ of a perfect graph G is perfect. Since $(\overline{\overline{G}}) = G$ this would complete the proof.

For $|G| = 1$, $\overline{G} = G$ so the statement is trivial.

Now let $|G| \geq 2$. Let $\mathcal{K}$ denote the set of all vertex sets of all complete subgraphs of G. Put $\alpha(G) =: \alpha$ and let $\mathcal{A}$ be the set of all stable sets A in G with $|A| = \alpha$.

Every $\boxed{\text{proper}}$ induced subgraph of $\overline{G}$ is the complement of a $\boxed{\text{proper}}$ induced subgraph of G and is hence perfect by induction. So it is enough to show $\chi(\overline{G}) \leq \omega(\overline{G}) = \alpha$

$H \triangleleft G$, $H \neq G$

$H_1 = G[V(H)] = H$, $H_1 := G[V(H)]$

To this end we will find a set $K \in \mathcal{K}$ such that $K \cap A \neq \emptyset$ $\forall A \in \mathcal{A}$; then $\omega(\overline{G} - K) = \alpha(G - K) < \alpha = \omega(\overline{G})$ (**)

and by induction hypothesis we have

$$\chi(\bar G) \le \chi(\bar G - k) + 1 \underset{\text{Ind. hyp.}}{=} \omega(\bar G - k) + 1 \underset{(**)}{\le} \omega(\bar G)$$

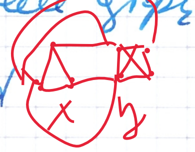
as desired.

So find a k as above. By contradiction!

Suppose now $\nexists\, k \in \mathcal{K}$ s.th. $k \cap A \neq \emptyset$, $\forall A \in \mathcal{A}$. $k(x)=3$, $k(y)=4$

i.e. $\forall k \in \mathcal{K}\ \exists A_k \in \mathcal{A}$, s.th. $k \cap A_k = \emptyset$.

Let us replace in G every vertex x by a complete graph G_x of order $k(x) := |\{k \in \mathcal{K} : x \in A_k\}|$, and joining all the vertices of G_x to all vertices of G_y whenever $\{x,y\} \in E(G)$. The graph G' obtained by this operation has vertex set $\bigcup_{x \in V} V(G_x)$ and two vertices $v \in G_x$, $w \in G_y$ are adjacent in G' iff $\{x,y\} \in E(G)$. Moreover G' can be obtained by repeated vertex expansion from $G[\{x \in V : k(x) > 0\}]$. Since this latter graph is an induced subgraph of G, it is perfect by assumption. So G' is perfect by the previous lemma. Thus $\chi(G') \le \omega(G')$ (**)



Next we will obtain a contradiction to (**) by computing the values of $\omega(G')$ and $\chi(G')$. By construction of G' every maximal compl. subgraph of G' has the form

$$G'\Big[\bigcup_{x \in X} G_x\Big]$$ for some $X \in \mathcal{K}$. So $\exists$ a set $X \in \mathcal{K}$

with $$\omega(G') = \sum_{x \in X} k(x) = |\{(x,k) : x \in X,\ k \in \mathcal{K},\ x \in A_k\}|$$

$$= \sum_{k \in \mathcal{K}} |X \cap A_k| \le |\mathcal{K}| - 1 \quad (***)$$ where the last inequality follows from the fact that $|X \cap A_k| \le 1\ \forall k \in \mathcal{K}$ (since A_k is indep. but $G[X]$ is complete)

and $|X \cap A_X| = \emptyset$ by the choice of A_X.

On the other hand

$$|G'| = \sum_{x\in V} k(x) = |\{(x,k) : x \in V,\ k \in \mathcal{K},\ x \in A_k\}|$$

$$= \sum_{k\in\mathcal{K}} \underbrace{|A_k|}_{|V \cap A_k|} = |\mathcal{K}| \cdot \alpha. \qquad (****)$$

Observe now that $\alpha(G') \leq \alpha$ by the construction of G'.

Thus $$\underbrace{\chi(G') \geq \frac{|G'|}{\alpha(G')}}_{\text{trivial bound}} \geq \frac{|G'|}{\alpha} \underbrace{=}_{(****)} |\mathcal{K}| \qquad (*****)$$

Putting together $(***)$ and $(*****)$ we get

$$\chi(G') \geq |\mathcal{K}| > |\mathcal{K}| - 1 \geq \omega(G')$$ in contradiction to $(**)$

$$\chi(G') \leq \omega(G')$$

$\blacksquare$