(Chapter 6) Perfect graphs: definitions and elementary concepts

Definition 1.

(Shannon 1956, Berge 1961)
A **graph** *G is called* **perfect** *iff $\chi(H) = \omega(H)$ holds for every induced subgraph H of G.*

Trivial example: bipartite graphs.

Definition 2.

A **property** *P of a graph G is called* **hereditary** *if every induced subgraph of G possesses property P.*

Observation: Being bipartite and being perfect are hereditary properties.

Proposition 1.

Complements of bipartite graphs are perfect graphs.

Proposition 2.

Line graphs of bipartite graphs are perfect graphs.

Definition 3.

An **edge cover (EC)** *in a graph* $G = (V, E)$ *is a subset* E_1 *of edges such that every vertex of* G *is incident with at least one edge in* E_1*, i.e.* $\forall v \in V$, $\exists e \in E_1$*, such that* $v \in e$*. The* **edge covering number of** G, $\boldsymbol{\rho(G)}$*, is the smallest cardinality of an EC in* G*:*
$\rho(G) = \min\{k \in \mathbb{N} : \exists \text{ edge cover } E_1 \subseteq E \text{ with } |E_1| = k\}$.

Definition 4.

A **vertex cover (VC)** *in a graph* $G = (V, E)$ *is a subset* V_1 *of vertices such that every edge of* G *is incident with at least one vertex in* V_1*, i.e.* $\forall e \in E$, $\exists v \in V_1$*, such that* $v \in e$*. The* **vertex covering number of** G*, denoted by* $\boldsymbol{\tau(G)}$*, is the smallest cardinality of a VC in* G*:*
$\tau(G) = \min\{k \in \mathbb{N} : \exists \text{ vertex cover } V_1 \subseteq V \text{ with } |V_1| = k\}$.

Observation: For general graphs G

- $\rho(G)$ can be determined in polynomial time,
- the (trivial) inequality $\rho(G) \geq \frac{|V|}{2} \geq \nu(G)$ holds,
- it is NP hard to determine $\tau(G)$,
- the (trivial) inequality $\tau(G) \geq \nu(G)$ holds.

Theorem 3.

(Gallai 1959)
For every graph G with $n = |G| = |V|$ the equality $\tau(G) + \alpha(G) = n$ holds. If G has no isolated vertices then $\rho(G) + \nu(G) = n$ holds.

Theorem 4.

(König 1931)
For every bipartite graph G the equality $\tau(G) = \nu(G)$ holds.

Definition 5.

The **clique pratition number** *of a graph $G = (V, E)$, $\boldsymbol{\theta(G)}$, is the smallest cardinality of a partition $\{V_i \colon i \in I\}$ of V for which $\forall i \in I$, $G[V_i]$ is a clique, i.e. $\theta(G) =$*

$$\min\{k \in \mathbb{N} \colon \exists (V_i)_{1 \leq i \leq k} \text{ with } V = \dot\bigcup_{i=1}^{k} V_i \text{ and } G[V_i] \text{ is a clique } \forall i \in \overline{1, k}.\}$$

Observation: $\theta(G) = \chi(\bar{G})$ holds for every graph G and its complement $\bar{G}$.

(Chapter 6) A characterisation of perfect graphs and strong perfect graphs

Lemma 5.

A graph G is perfect if and only if every induced subgraph H of G contains a stable set $S(H)$ which has a nonempty intersection with every maximum clique of H.

Definition 6.

(Berge, Duchet 1984)
A graph G is called **strongly perfect** *if every subgraph H of G contains a stable set $S(H)$ which has a nonempty intersection with every maximum clique of H.*

Corollary 6.

Strongly perfect graphs are perfect graphs.

The converse is not true, i.e. there are (nontrivial) perfect graphs which are not strongly perfect.

(Chapter 6) Some optimization problems on perfect graphs

Theorem 7.

(Grötschel, Lovász, Schrojver 1981)
For perfect graphs the numbers $\chi(G)$, $\alpha(G)$, $\omega(G)$, and $\theta(G)$ as well an optimal coloring, a maximum stable set, a maximum clique and an optimal cliqe partition can be computed in polynomial time.

These polynomial time algorithms for general perfect graphs make use of the **ellipsoid method** from the linear programming.
For particular subclasses of perfect graphs, e.g. bipartite graphs, line graphs of bipartite graphs, chordal graphs, **combinatorial algorithms** are known.

(Chapter 6) The recognition problem for perfect graphs

The perfect graph recognition problem
Input: A graph $G = (V, E)$
Question: Is G perfect?

Theorem 8.

(Lovász perfect graph theorem, 1972;
also known as **Berge's weak perfect graph conjecture**, *1961)*

A graph is perfect if and only if its complement is perfect.

Theorem 9.

(The strong perfect graphs theorem, Chudnovsky, Robertson, Seymour and Thomas 2006;
also known as **Berge's strong perfect graph conjecture**, *1963))*

A graph is perfect if and only if neither G nor its complement $\bar{G}$ contain an odd cycle of length at least 5 as an induced subgraph.