Hamiltonian cycles and Hamiltonian graphs

Definition 1.

A cycle C in a graph $G = (V, E)$ is called a Hamiltonian cycle if $V(C) = V(G)$. A Hamiltonian graph is a graph which contains a Hamiltonian cycle (as a subgraph).

Hamiltonian Cycle Problem (HCP)
Input: A graph $G = (V, E)$
Question: Is G Hamiltonian?

Proposition 1.

(Karp 1972)
HCP is NP-complete.

There are a couple of sufficient conditions (SC) which imply the hamiltonicity of a given graph (i.e. the property of being Hamiltonian). The are are also a few necessary conditions (NC). Most of the NC are not deep results.

Lemma 1.

Let G be a Hamiltonian graph. Then $G - S$ contains at most $|S|$ components, for any S with $\emptyset \subsetneq S \subsetneq V(G)$

Proof: homework, trivial.

Hamiltonicity: sufficient conditions

Lemma 2.

(Bondy, Chvátal 1976)
Let G be a graph with $|G| \geq 3$ and let $u, v \in V(G)$ with $\{u, v\} \notin E(G)$ and $deg(u) + deg(v) \geq |G|$. Then the following holds:
G is Hamiltonian iff $G + \{u, v\}$ is Hamiltonian.

Definition 2.

For $k \in \mathbb{N}$, the k-th Hamiltonian hull $H_k(G)$ of a graph G is recursively defined as follows: if there are non-adjacent vertices $u, v \in V(G)$ with $deg(u) + deg(v) \geq k$, then set $H_k(G) := H_k(G + \{u, v\})$ otherwise set $H_k(G) := G$.

Lemma 3.

The k-Hamiltonian hull $H_k(G)$ of a graph G is well defined.

Theorem 1.

(Bondy, Chvátal 1976, a corollary of Lemma 2)
A graph G with $n := |G|$ is Hamiltonian iff the n-th Hamiltonian hull $H_n(G)$ is Hamiltonian.

Corollary 2.

(Ore 1960)
If in a graph G with $n := |G|$, $n \geq 3$, $deg(u) + deg(v) \geq n$ holds for any $u, v \in V(G)$ with $\{u, v\} \notin E(G)$, then G is Hamiltonian.

Corollary 3.

(Dirac 1952)
A graph G with $n := |G| \geq 3$ and $\delta(G) \geq \frac{n}{2}$ is Hamiltonian. This bound is best possible.

Theorem 4.

(Chvátal, Erdös 1972)
Consider a graph G with $n := |G| \geq 3$. The inequality $\kappa(G) \geq \alpha(G)$ implies the hamiltonicity of G.
($\alpha(G)$ is the stability number of G, i.e.
$\alpha(G) := \max\{|S| : S \subseteq V(G)$ and $G[S]$ has no edges$\}$.)