Hamiltonicity

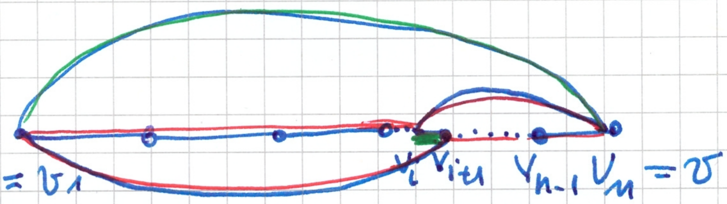
Lemma 2 (Chvátal 1976, Bondy)

Let G be a graph with $|G| \geq 2$.
Let $v, u \in V(G)$ with $\{u, v\} \notin E(G)$ and $\deg(u) + \deg(v) \geq |G|$.

Then G is Hamiltonian $\iff$ $G + \{u, v\}$ is Hamiltonian

Proof $\Rightarrow$ trivial (adding an edge to a Ham. graph does not destroy hamiltonicity)

$\Leftarrow$ Assume $G + \{u, v\}$ is Hamiltonian and let C be a Hamiltonian cycle (HC) in $G + \{u, v\}$.
If $\{u, v\} \notin E(C)$, then C is also a HC in G.
So assume w.l.o.g. $\{u, v\} \in E(C)$ and
let $C = \{u = v_1, \dots v_n = v\}$



Assume $\exists\ i \in \{2, \dots, n-2\}$ s.that $e_R := \{u, v_{i+1}\} \in E(G)$
and $e_L := \{v_i, v\} \in E(G)$.

Then $C' := \big(C \cup \underline{\{\{u, v_{i+1}\}, \{v_{i-1}, v\}\}}\big) \setminus \underline{\{\{u, v\}, \{v_i v_{i+1}\}\}}$

is a HC in G.

So we only need to show that such a 'cross-over' exists

Let $R := \{2 \leq i \leq n-2 : \{u v_{i+1}\} \in E(G)\}$ and
$L := \{2 \leq i \leq n-2 : \{v_i, v\} \in E(G)\}$

Since $L \cup R \subseteq \{2, \dots, n-2\}$ we have $|L \cup R| \leq n-3$

On the other hand (2)

$L = \{j : v_j \in N_G(v)\} \setminus \{n-1\}$ and $R = \{j : v_{j+1} \in N_G(u)\} \setminus \{1\}$

Thus $|R| + |L| = \deg_G(u) - 1 + \deg_G(v) - 1 \geq n-2 > |L \cup R|$

This can only happen if $R \cap L \neq \emptyset$
(because $|L \cup R| = |R| + |L| - |L \cap R|$)

so $\exists$ $i \in \{2, \dots, n-2\}$ with $\{u, v_{i+1}\} \in E$ $\{v_i, v\} \in E$
and this is the required cross-over.

$\boxtimes$

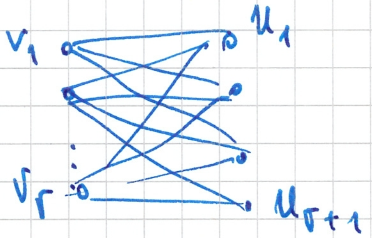
Corollary 3 (Dirac 1952)

Let G be a graph with $n := |G| \geq 3$. and $\delta(G) \geq \frac{n}{2}$.
Then G is Hamiltonian.

Proof $\forall$ $u, v \in V(G)$ with $\{u, v\} \notin E(G)$
we have $\deg(v) + \deg(u) \geq \delta(G) + \delta(G) \geq 2 \cdot \frac{n}{2} = n$

Corollary 2 $\Longrightarrow$ G is Hamiltonian.

The result is best possible:
Consider for example $K_{r, r+1}$ (complete bipartite with vertices $\{v_1, \dots v_r, u_1, \dots u_{r+1}\}$



and edges
$\{v_i, u_j\}$ $\forall$ $i \in \overline{1,r}$, $\forall j \in \overline{1, r+1}$

Then $\delta(K_{r,r+1}) = r = \frac{n-1}{2}$ where $n = |K_{r,r+1}| = 2r+1$

The graph $K_{r,r+1}$ is not Hamiltonian because it does not contain odd cycles and hence it cannot contain a cycle on $2r+1 = n$ vertices. $\boxtimes$

Theorem 4 (Erdös, Chvátal 1972)

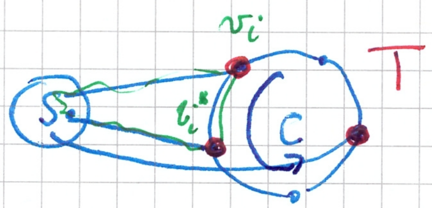
Let G be a graph with $n = |G| \geq 3$.
Then $\kappa(G) \geq \alpha(G)$ implies "G is Hamiltonian".

Proof: If $\alpha(G) = 1$ then $G = K_n$ ✓

Assume w.l.o.g. $\kappa(G) \geq \alpha(G) \geq 2$ and let C be a cycle in G. If $|C| < n$, then let S be a conn. component of $G[V \setminus V(C)]$ and let

$$T = V(C) \cap \left(\bigcup_{s \in S} N(s) \right)$$

be the set of vertices in C which have neighbors in S.
T is then a separator and $\kappa(G) \geq 2$ implies $|T| \geq 2$.



Since $G[S]$ connected we have:

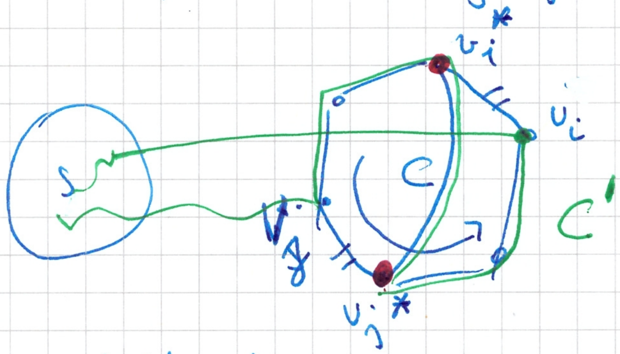
$\forall\, v_1, v_2 \in T \;\exists\; v_1 - v_2$-path P with inner vertices in S.

Let $C = (v_1, v_2, \dots, v_\ell)$ and let us denote by v^* the direct successor of v_i in an arbitrary but fixed orientation of the cycle
If for some $v_i \in T$ also $v_i^* \in T$ then Cycle C can be extended to a longer one by substituting the edge $(v_i v_i^*)$ by the $v_i - v_i^*$-path with inner vertices in S (Extension 1)

This holds also if $\{v_i^*, v_j^*\} \in E$ for some $v_i, v_j \in T$:

In this case we extend C to C' where

$$C' = C \setminus \{\{v_i\, v_i^*\}, \{v_j, v_j^*\}\} \cup \{\{v_i^*\, v_j^*\} \cup v_i^*\text{-}v_j^*\text{-path}\}$$



and $|C'| > |C|$ holds. (Extention 2)

Claim

The assumption $\kappa(G) \geq \alpha(G)$ implies that there always will be such 2 vertices as above as long as $|C| < n$.

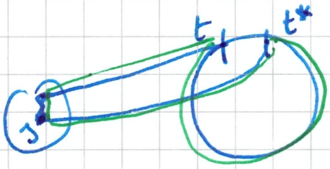
Proof of the claim

Indeed let $T^* := \{t^* : t \in T\}$

Let $|C| < n$ hold and let us assume that C cannot be further extended as above.

Then we get $\{s, t^*\} \notin E \quad \forall s \in S \ \forall t \in T$

because otherwise C would be extended (Extention 1)



$\Rightarrow |T^*| = |T|$ and T^* is stable, (otherwise Extention 2)

But also $T^* \cup \{s\}$ is stable $\forall s \in S$ and this leads to a contradiction

$$\Rightarrow \kappa(G) \leq |T| = |T^*| < |T^* \cup \{s\}| \leq \alpha(G) \ ↯$$

End of the proof of the claim

Now pick the largest cycle C in G.

If $|C| < n$ we can apply the claim, thus extend C, which was the largest cycle ↯