Advanced and algorithmic graph theory
TU Graz, summer term 2020

General definitions and notations

Definition 1 **An undirected graph** *(often also simply called* **graph***) G is an ordered pair (V,E), where V is the so-called* **set of vertices** *and $E \subseteq \{\{u,v\}: u,v \in V, u \neq v\} =: \mathcal{P}_2(V)$ is the* **set of edges***. Thus the edges of a graph are two-element subsets of the set of vertices of the graph. Given a set V we denote by $\mathcal{P}_2(V)$ as above the set of two-element subsets of V.*

A directed graph*, or briefly* **digraph***, G is a pair (V,E), where V is the so-called* **set of vertices** *and $E \subseteq V \times V$ is the* **set of arcs***. For a given graph (digraph) G we denote by $V(G)$ and $E(G)$ its set of vertices and its set of edges (arcs), respectively.*

The order of a graph (digraph) G is the cardinality $|V(G)|$ of its vertex set $V(G)$.

An edge e of a graph is denoted by $e := \{u,v\}$ and an arc of a digraph is denoted by (u,v); u and v are called **end-vertices** *of e. and We say that vertex u (v) is* **incident** *with edge (arc) e and that the vertices u and v are* **adjacent vertices** *or* **neighbors***. Moreover we say that e is an edge at u (v) or that arc e starts at u and ends at v respectively. Two edges e, f are called* **adjacent edges** *if $e \neq f$ and $e \cap f \neq \emptyset$. The set of* **neighbors of a vertex** *v in G is denoted by $N_G(v)$, or briefly by $N(v)$, if there is no ambiguity about the underlying graph G. Give a set $U \subseteq V(G)$ we define $N(U)$ to be the set of the* **neighbors of** *U by $N(U) := (\cup_{v \in U} N(v)) \setminus U$.*

The **degree** *$deg_G(v)$* **(or** *$deg(v)$***) of some vertex** *v* **in a graph** *G is defined as $deg(v) := |N(v)|$. A vertex with degree equal to 0 is called an* **isolated vertex** *in G. The* **minimum degree** *$\delta(G)$* **of a graph** *G is defined as $\delta(G) := \min_{v \in V(G)} deg(v)$. Analogoulsy, the* **maximum degree** *$\Delta(G)$* **of a graph** *G is defined as $\Delta(G) := \max_{v \in V(G)} deg(v)$. The* **average degree** *$d(G)$* **of a graph** *G is defined as $d(G) := \frac{1}{|V(G)|}\sum_{v \in V(G)} deg(v)$. The* **density** *$\varepsilon(G)$* **of a graph** *$G$ is defined as $\varepsilon(G) := \frac{|E(G)|}{|V(G)|}$.*

Question 1 *Recall the handshake lemma and derive a simple relationship between $d(G)$ and $\varepsilon(G)$.*

*For some $k \in \mathbb{Z}_+$, a k-***regular graph** *G is a graph with $deg(v) = k$, for all $v \in V(G)$. A 3-regular graph is also called a* **cubic graph***.*

A graph G is called **a complete graph** *if all vertices in $V(G)$ are pairwise adjacent. The complete graph of order n is denoted by K_n (for some $n \in \mathbb{N}$).*

A set $A \subseteq V(G)$ is called an **independent set** *in G if no two vertices of A are adjacent. The* **independence number of a graph** *G is denoted by $\alpha(G)$ and is defined as the largest $k \in \mathbb{N}$ such that there exists an independent set of cardinality k in G, i.e.*

$$\alpha(G) := \max\{|A|: A \subseteq V(G), A \text{ is an independent set in } G\}.$$

Definition 2 *G and G' are called* **isomorphic graphs** *iff there eists a bijection $\phi: V(G) \to V(G')$ such that $\{u,v\} \in E(G) \iff \{\phi(u),\phi(v)\} \in E(G')$. In this case ϕ is called an* **isomorphism** *of G and G'. If G and G' are isomporphic we denote $G \simeq G'$. Usually we make no difference between two isomorphic graphs and denote $G = G'$ instead of $G \simeq G'$. If $G = G'$ then an isomorphism ϕ of G and G' is called an* **automorphism***.*

A class of graphs which is closed under isomorphism is called **a graph property**. *For example, containing a triangle is a graph property. A* **triangle in a graph** G *is a triple of pairwise distinct vertices* $v_1, v_2, v_3 \in V(G)$ *such that any two of them are adjacent.*

Question 2 *Can you specify other graph properties?*

A mapping taking graphs as arguments is called **a graph invariant** *iff it assigns equal images (values) to isomorphic graphs. For example the number of vertices and the number of edges in a graph are graph invariantes.*

Question 3 *Can you specify other graph invariants?*

Definition 3 *Consider two graphs* $G = (V, E)$ *and* $G' = (V', E')$. *The* **union of** G **and** G' *is defined as* $G \cup G' := (V \cup V', E \cup E')$, **the intersection of** G **and** G' *is defined as* $G \cap G' := (V \cap V', E \cap E')$.
The graphs G *and* G' *are called* **disjoint graphs** *if* $V(G) \cap V(G') = \emptyset$. *Given two disjoint graphs* G *and* G' *their product* $G * G'$ *is obtained from* $G \cup G'$ *by adding to it the edges of the form* $\{v, v'\}$ *for all* $v \in V(G)$, $v' \in V(G')$.

Question 4 *Draw the graph* $K_2 * K_3$*? Is this product a complete graph?*

The **complement** G^C **of a graph** G *is defined as* $G^C := \big(V(G), \mathcal{P}_2(V(G)) \setminus E(G)\big)$. *The* **line graph** $L(G)$ **of a graph** G *is defined as* $L(G) := \big(E(G), \big\{\{e, f\} \colon e, f \in E(G), e \cap f \neq \emptyset\big\}\big)$.
If $V \subseteq V'$ *and* $E \subseteq E'$ *we say that* G **is a subgraph of** G' *or* G' **is a supergraph of** G *and denote* $G \subseteq G'$. *If* $G \subseteq G'$ *and* $G \neq G'$ *we say that* G **is a proper subgraph of** G' *and denote* $G \subsetneq G'$.
If $G \subseteq G'$ *and* $E(G)$ *contains all edges* $\{u, v\}$ *of* $E(G')$ *with* $u \in V(G)$ *and* $v \in V(G)$ *we call* G *an* **induced subgraph** *of* G'. *In this case we say that* V **induces** G **in** G' *and denote* $G = G'[V]$.

Question 5 *Given that* G *is an induced subgraph of* G'. *Does the following equality hold*

$$E(G) = \big\{\{u, v\} \colon u \in V, v \in V\big\} \cap E(G')\,?$$

A subgraph G *is called* **a spanning subgraph** *of* G' *iff* $G \subseteq G'$ *and* $V(G) = V(G')$.

Definition 4 *Given a graph* G *and a set* $U \subseteq V(G)$ *we write* $G - U := G[V \setminus U]$. *If* U *is a* **singleton**, *i.e.* $U = \{v\}$ *for some* $v \in V(G)$ *we write* $G - v$ *instead of* $G - \{v\}$. *We also write* $G - G'$ *instead of* $G - V(G')$. *For some* $F \subseteq \mathcal{P}_2(V(G))$ *we write* $G - F := (V(G), E(G) \setminus F)$ *and* $G + F := (V(G), E(G) \cup F)$. *If* F *is a singleton, i.e.* $F = \{e\}$ *for some* $e \in \mathcal{P}_2(V(G))$, *we write* $G + e$ *and* $G - e$ *instead of* $G + \{e\}$ *and* $G - \{e\}$, *respectively.*

Definition 5 *A graph* G *is called* **edge-maximal** *with respect to some graph property* P *iff* G *itself has the property* P, *but the graph* $G + \{u, v\}$ *does not have property* P, *for all* $u, v \in V(G)$ *such that* $\{u, v\} \notin E(G)$. *If we say a* **graph** G **is be maximal (minimal) with respect to some property** P *without any further specification, then we mean that* G *has the property* P *but no supergraph (subgraph) of* G *has it. Analogosly, if we speak of a* **maximal or minimal set of vertices of edges** *the considered order relation is simply the set-inclusion.*

Question 6 *Let P be the property "contains a triangle" and let P' the property "does not contain a triangle" or equivalently "is triangle-free"*[1]. *Give a graph of order 6 which is minimal with respect to P. Give a graph of order 6 which maximal with respect to P'*

Definition 6 *A* **path** *P is a graph $P = (V, E)$ with $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \big\{\{x_{i-1}, x_i\} \colon i \in \{1, 2, \ldots, k\}\big\}$, where all x_i, $i \in \{0, 1, \ldots, k\}$, are pairwise disjoint and $k \in \mathbb{Z}_+$. Analogoulsy, a* **directed path (dipath)** *P is defined as a digraph $P = (V, E)$ with $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \big\{(x_{i-1}, x_i) \colon i \in \{1, 2, \ldots, k\}\big\}$, where all x_i, $i \in \{0, 1, \ldots, k\}$, are pairwise disjoint. The vertices $x_1, \ldots, x_{k-1}$ are called* **internal vertices of the path (dipath)** *P, x_0 and x_k are called* **start-vertex of** *P and* **end-vertex of** *P, respectively, or* **end-vertices of** *P commonly. We say that the end-vertices of path P are joined (connected) by P and P joins (connects) its end-vertices. The* **length of the path (dipath)** *is the number k of its edges (arcs). Often a path (dipath) of length k is denoted by P^k. Thus a path (dipath) of length 0 is just a single vertex (or the complete graph K_1; such a path (dipath) is called a* **trivial path** *(***trivial dipath***). Often we refer to the path (dipath) P as above by the natural sequence of its vertices and denote $P = x_0, x_1, \ldots, x_k$. Let $P = x_0, x_1, \ldots, x_k$. We distinguish the following* **subpaths** *of P:*
$Px_i := x_0, x_1, \ldots, x_i$, $x_iP = x_i, x_{i+1}, \ldots, x_k$, $x_iPx_j := x_i, x_{i+1}, \ldots, x_{j-1}, x_j$, and $\mathring{P} = x_1, \ldots, x_{k-1}$, for $i, j \in \{0, 1, \ldots, k\}$.

Given a graph G and a path P we say that P is a path in G iff $P \subseteq G$, i.e. if the path P is a subgraph of G.

A **walk** *W* **of length** *k*, *$k \in \mathbb{N}$*, **in a graph** *G is a non-empty alternating sequence $v_0.e_0, v_1, e_1, \ldots, e_{k-1}, v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\} \in E(G)$, for all $i \in \{0, 1, \ldots, k-1\}$. If $v_0 = v_k$ the walk is called a* **closed walk***. If the vertices of the walk W are pairwise distinct, then W defines a path in G.*

Consider a graph G and two subsets $A, B \subseteq V(G)$. A path $x_0, x_1, \ldots, x_k$ in G is called **an** *A-B-***path** *if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$*[2]*. If $A = \{a\}$ and/or $B = \{b\}$ we write a-B-path, A-b-path or a-b-path, respectively. Two or more paths are* **independent paths** *iff none of them contains an inner vertex of another.*

Given a graph H we call a path $P = x_0, \ldots, x_k$ an H-**path** *if P is non-trivial (i.e. $k \geq 1$) and $V(P) \cap V(H) = \{x_0, x_k\}$.*

If $P = x_0, \ldots, x_{k-1}$ is a path and $k \geq 3$, then $C := P + \{x_{k-1}, x_0\}$ is called a **cycle***. Sometimes we denote $C = x_0, x_1, \ldots, x_{k-1}, x_0$. The* **length of a cycle** *is the number of its edges. A cycle of length k is often denoted by C^k. A* **cycle in a graph** *G is a subgraph of G which is a cycle. The minimum length of a cycle in a graph G is called the* **girth of** *G and is denoted by $g(G)$. The maximum length of a cycle in a graph G is called the* **circumference of** *G and is denoted by $circ(G)$. If G contains no cycle we set $g(G) = \infty$ and $circ(G) = 0$. An edge joining two vertices of a cycle C which is not an edge of C is called a* **chord of the cycle** *C. Thus an induced cycle in G (i.e. a cycle which is an induced subgraph of G) is a cycle without chords.*

[1] Convince yourself that P' is indeed a property!

[2] This concept and the further concepts within this definition can be analogously defined for digraphs and dipaths.

Proposition 1 *Every graph G contains a path of length $\delta(G)$ and a cycle of length $\delta(G)+1$, provided that $\delta(G) \geq 2$.*[3]

Definition 7 *The* **distance** $d_G(u, v)$ **(or** $d(u, v)$**) of two vertices** *u and v in a graph G is the length of a shortest u-v-path in G; if there is no such a path we set $d_G(u, v) = \infty$.*

The **diameter of** *G is denoted by $diam(G)$ and is defined as $diam(G) := \max_{u,v \in V(G)} d_G(u, v)$.*

Given a graph G, a vertex $v \in v(G)$ is called a **central vertex** *if its greatest distance form any other vertex is as small as possible. This distance is called the* **radius of graph** *G and is denoted by $rad(G)$, i.e. $rad(G) := \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$.*

Question 7 *Determine $diam(C^k)r$ and $rad(C^k)$ of a cycle with k vertices, $k \in \mathbb{Z}$, $k \geq 3$. Can a graph have more than one central vertex? Do the inequalities $rad(G) \leq diam(G) \leq 2rad(G)$ hold? Are these inequalitioes best possible. i.e. are there particular graphs for which these inequalities are fulfilled with equality?*

Definition 8 *A graph G with at least one vertex is called* **connected** *iff for any two vertices $u, v \in V(G)$ there esists a u-v-path in G. Otherwise G is called* **disconnected***. A set $U \subseteq V(G)$ is called a* **connected set of vertices in** *G iff $G[U]$ is connected.*

G is called a **k-connected graph** *iff $|G| > k$ and $G - X$ is connected, for all $X \subset V$ with $|X| < k$. The largest nonnegative integer k such that G is k-connected is called* **the connectivity of graph** *G and is denoted by $\kappa(G)$.*

Question 8 *Which graphs are 0-connected? Does 1-connected mean conected? For which graph does $\kappa(G) = 0$ hold? Specify a graph G with $|G| = n$ and $\kappa(G) = n - 1$, for $n \in \mathbb{N}$.*

If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E(G)$ of fewer than ℓ vertices then G is called an **ℓ-edge connected graph***. The largest nonnegative integer ℓ such that G is ℓ-edge connected is called the* **edge connectivity of graph** *G and is denoted by $\lambda(G)$.*

Question 9 *Specify $\lambda(G)$ for a disconnected graph G. Specify a 2-edge connected graph G with $|G| = n$, for $n \in \mathbb{N}$.*

Definition 9 *A maximal connected subgraph of a graph G is a* **component of graph** *G. By definition a graph without vertices has no components.*

Question 10 *Are the components of G induced subgraphs of G? Do their vertex sets partition $V(G)$?*

If $A, B \subseteq V(G)$ and $X \subseteq V \cup E$ are such that every A-B-path in G contains a vertex or an edge from X, we say that X **separates the sets** *A* **and** *B* **in** *G. Notice that this implies $A \cap B \subseteq X$.*

We say that $X \subset V(G) \cap E(G)$ **separates** *G if $G - X$ is disconnected, i.e. if X separates in G some two vertices that are not in X. A separating set of vertices is called a* **separator***. Separator sets of edges have no generic name, but some of them do (e.g. cuts and bonds, to be defined later).*

A vertex which separates two other vertics from the same component is called a **cutvertex***. An edge which separates its own end-vertices is called a* **bridge***.*

[3] In contrast to this statement, the minimum degree and the girth are not related to each other.

Question 11 *Draw some examples of cuts and bridges. Can you characterize bridges by means of cycles?*

Definition 10 *A graph G is called an* **acyclic graph** *or a* **forest** *iff it contains no cycles. A connected forest is called a* **tree**. *A vertex of degree one in a tree is called a* **leaf of the tree**. *A* **rooted tree** *is a tree T with a special vertex r in it, $r \in V(T)$, called* **root of the tree**. *Given a rooted tree T with root r* **the tree-order** *is a partial order $\leq$ on $V(T)$ such that $x \leq y$ iff x lies on the unique r-y-path in T, for all $x, y \in V(T)$.*

We think of the tree-order as expressing **height**, *i.e. if $x < y$ (which means that $x \leq y$ and $x \neq y$ hold), we say that x* **lies below** y **in** T. *We denote by $\lceil y \rceil := \{x \in V(T) \colon x \leq y\}$ the* **down-closure of** y; *analogously, we denote by $\lfloor x \rfloor := \{y \in V(T) \colon y \geq x\}$ the* **up-closure of** x. *A set $X \subseteq V(T)$ that equals its up-closure, i.e. which satisfies $X = \lfloor X \rfloor := \cup_{x \in X} \lfloor x \rfloor$, is said to be* **closed-upwards** *or an* **up-set**. *A set which is* **closed-downwards** *or a* **down-set** *is defined analogously. The vetices at distance k from the root are said to* **have height equal to** k *and form the* k**-th level of** T.

Question 12 *Given a tree T with root $r \in V(T)$. Are the end-vertices of any edge $\{x.y\}$ of T* **comparable** *in terms of the tree-order, i.e. does $x \leq y$ or $y \leq x$ hold, for all $e = \{x, y\} \in E(T)$? Is the down-closure of every vertex a* **chain**, *i.e. a set of pairwise comparable elements?*

A rooted tree T with root r contained in a graph G is called a **normal tree in** G *iff all the end-vertices of every T-path in G are comparable in the tree-order of T.* **Normal spanning trees of a graph** G *are the so called* **depth first search trees of** G *(because of the way they arise in computer searches on graphs).*

Question 13 *Give the example of a graph G with $|G| = 15$ and a normal spanning tree in G. Observe that the following statement holds*

Proposition 2 *Every connected graph contains a normal spanning tree, with any specified vertex as its root.*