Theorem 1.

(Menger 1927)

(a) (**The edge version**) *Let s and t be two arbitrary vertices of a graph G. The maximum number of s-t-paths which do not share an edge equals the minimum cardinality of a set of edges which separates s and t in G*

(b) (**The vertex version**) *Let s and t be arbitrary vertices of a graph G such that* $\{s, t\} \notin E(G)$. *The maximum number of independent s-t-paths equals the minimum cardinality of an s-t-separator G.*

Proposition 1.

Menger's Theorem implies an equivalent definition of connectivity:

(a) (**Edge connectivity**) *A graph G with* $|G| > 1$ *is* ℓ*-edge connected,* $\ell \in \mathbb{N}$, *iff (i) every edge set which separates G has at least* ℓ *elements, or equivalently (ii)* $\forall s, t \in V(G)$ *there exists* ℓ *edge-disjoint s-t-paths.*

(b) (**Vertex connectivity**) *A graph G with* $|G| > k$, $k \in \mathbb{N}$, *is k-edge connected iff (i) every separater of G has at least k elements, or equivalently (ii)* $\forall s, t \in V(G)$ *there exists k independent s-t-paths.*

Theorem 2.

If the graph G is non-trivial, i.e. $|G| > 1$, the $\kappa(G) \leq \lambda(G) \leq \delta(G)$ holds.

Thus high connectivity implies high minimum degree.

The converse is not true: there are graphs with large $\delta(G)$ which have a small $\lambda(G)$. However a high $\delta(G)$ implies the existence of a highly connected subgraph.

Proposition 2.

(Mader 1972)
Let $k \in \mathbb{N}$, $k \neq 0$. Every graph G with $d(G) \geq 4k$ has a (k+1) connected subgraph H such that $\epsilon(H) = \frac{|E(H)|}{|V(H)|} > \epsilon(G) - k$ holds.

Observation 3.

The vertices of a connected graph G can always be enumerated as $v_1, v_2, \ldots, v_{|G|}$ such that the induced subgraph $G_i := G[\{v_1, v_2, \ldots, v_i\}]$ is connected, for all $i \in \overline{1, |G|}$.

(The proof is a homework.)

2-Connectivity and 3-Connectivity

Definition 1.

A maximal connected subgraph without a cut vertex is called a block.

Observation 4.

Every block is either (i) a maximal 2-connected subgraph, or (ii) a bridge, or (iii) an isolated vertex. Conversely, every such graph is a block.

Observation 5.

(a) Different blocks of a graph G overlap in at most one vertex which is then a cut-vertex, (b) every edge of G lies in a unique block, and (c) G is the union of its blocks.

Definition 2.

The block-cut-vertex graph $bc(G)$ of a given graph G is a graph with vertex set $\mathcal{B} \cup \mathcal{C}$, where $\mathcal{B}$ is the set of blocks of G and $\mathcal{C}$ is the set of cut-vertices of G. The edge set of $bc(G)$ consists of edges $\{b, c\}$ with $b \in \mathcal{B}$, $c \in \mathcal{C}$, and $c \in b$.

Theorem 6.

(Gallai 1964, Harary and Prins 1966)
The block-cut-vertex graph $bc(G)$ of a connected graph C is a tree.

Theorem 7.

(Ear decomposition, Whitney 1932, Halin and Jung 1963)
A graph G is 2-connected iff it can be represented as $G = C \cup P_1 \cup P_2 \cup \ldots \cup P_k$, for some $k \in \mathbb{N}$, where C is a cycle and P_i $1 \leq i \leq k$, are paths which have only their end-vertices in common with $C \cup P_1 \cup \ldots \cup P_{i-1}$.

Definition 3.

Contraction of an edge

Let $e = \{x, y\}$ be an edge of a graph $G = (V, E)$. We denote by G/e the graph obtained from G by contracting the edge e into a new vertex v_e which becomes adjacent to all the former neighbors of x and of y. Formally $G/e := (V', E')$, where

$$V' := V \setminus \{x, y\} \cup \{v_e\} \text{ and}$$

$$E' := \Big\{\{v, w\} \in E \colon \{v, w\} \cap \{x, y\} = \emptyset\Big\} \cup \Big\{\{v_e, u\} \colon \{x, u\} \in E \text{ or } (y, u\} \in E\Big\}.$$

More generally, if X is another graph and $\{V_X \colon x \in V(X)\}$ is a partition of V into connected subsets such that for any two vertices $x, y \in X$ there is a V_x-V_y-edge in G iff $\{x, y\} \in E(X)$, we call G an MX and write $G = MX$. The sets V_x are called the branch sets of this MX.

Intuitively , MX is obtained from G by contracting every branch set to a single vertex and deleting any parallel edges or loops that may arise.

If $MX = G$ is a subgraph of Y, $G \subseteq Y$, we call X a minor of Y and denote $X \preceq Y$.

Contraction of a set of vertices

If $V_x = U \subset V(G)$ is one of the branch sets and every other branch set consists just of a single vertex, we also write G/U for the graph X and V_U for the vertex $x \in V(X)$ to which U contracts and think of the rest of X as an induced subgraph of G. The contraction of edge $\{u, u'\}$ corresponds to the simple case $U = \{u, u'\}$.

Proposition 3.

G is an MX iff G can be obtained from G by a series of edge contractions, i.e. iff there exist an $n \in \mathbb{N}$, the graphs G_0, G_1, ..., G_n and the edges $e_i \in E(G_i)$ such that $G_0 = G$, $G_n \simeq X$ and $G_{i+1} = G_i/e_i$ for $0 \leq i \leq n-1$.

Proof: Induction on $|G| - |X|$, homework.

Lemma 1.

If G is 3-connected and $|G| > 4$, then G has an edge e such that G/e is again 3-connected.

Theorem 8.

(Tutte 1961) A graph G is 3-connected iff there exits an $n \in \mathbb{N}$ and sequence of graphs G_0, G_1, ..., G_n with the following properties:

(i) $G_0 = K_n$ *and* $G_n = G$

(ii) G_{i+1} *has an edge* $\{x, y\}$ *with* $deg_{G_{i+1}}(x) \geq 3$, $deg_{G_{i+1}}(y) \geq 3$, *and* $G_i = G_{i+1}/\{x, y\}$, *for all* $0 \leq i \leq n-1$.