(5) Edge coloring

Observations: A k-regular graph is a Class I graph iff it is 1-factorizable, i.e. if its edge set can be partitioned into k matchings.
A graph G with $|E(G)| > \Delta(G)\nu(G)$ is a Class II graph.

Corollary 12.

(Vizing 1964, Fournier 1973)
Let G be a graph and $S \subseteq V(G)$ the set of nodes with maximum degree in G, $S := \{v \in v(G) \colon deg(v) = \Delta(G)\}$. If the induced subgraph $G[S]$ is cycle-free, i.e. a forest, then G is a Class I graph. An edge coloring with $\Delta(G)$ colors can be constructed in $O(|E(G)||V(G)|)$ time.

Theorem 13.

(Frieze, Jackson, McDiarmid and Reed 1981)
Let p_n be the percentage of Class II graphs among graphs on n vertices with a fixed numbering. Then $\forall \epsilon > 0$ and n large enough

$$n^{-(\frac{1}{2}+\epsilon)n} < p_n < n^{-(\frac{1}{8}-\epsilon)n} \text{ holds.}$$

Thus almost all graphs are Class I graphs, graphs of Class II are extremely rare.

(5) Choosability and list coloring

Definition 6.

Let a graph $G = (V, E)$ and a list $L(v)$ of colors for every $v \in V$ be given. A **list coloring** *of G with lists $L(v)$, for $v \in E$, is a mapping $c\colon V \to \cup_{v \in V} L(v)$ such that $c(v) \in L(v)$ for all $v \in V$ and $c(v) \neq c(w)$ for all $\{v, w\} \in E$.*

*A graph G is called k-***choosable** *if for any given lists $L(v)$, $v \in V(G)$, which contain k pairwise different colors each, there exists an list coloring with lists $L(v)$, for $v \in V(G)$. The smallest natural number k for which G is k-choosable is the* **list chromatic number** *of G, $\chi_\ell(G)$.*

Observation: For any graph G the inequality $\chi_\ell(G) \geq \chi(G)$ holds. There are examples where the gap between the two numbers is large.

Theorem 14.

(Alon 1993)
There exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, given any integer k, all graphs G with average degree $d(G) \geq f(k)$, satisfy $\chi_\ell(G) \geq k$.

Theorem 15.

(Thomassen 1994)
Every planar graph is 5-choosable.

(5) Edge choosability and edge list coloring

Definition 7.

Let a graph $G = (V, E)$ and a list $L(e)$ of colors for every $e \in E$ be given. A **(edge) list coloring** *of G with lists $L(e)$, for $e \in E$, is a mapping $c\colon E \to \cup_{e \in E} L(e)$ such that $c(e) \in L(e)$ for all $e \in E$ and $c(e) \neq c(f)$ forall $e, f \in E$ with $e \cap f \neq \emptyset$.*

*A graph G is called k-***edge-choosable** *if for any given lists $L(e)$, $e \in E(G)$, which contain k pairwise different colors each, there exists an (edge) list coloring with lists $L(e)$, for $e \in E(G)$. The smallest natural number k for which G is k-edge-choosable is the* **list chromatic index** *of G, $\chi'_\ell(G)$.*

Observation: For any graph G the inequality $\chi'_\ell(G) \geq \chi'(G)$ holds.

A conjecture of Vizing: $\chi'_\ell(G) = \chi'(G)$ holds for any graph G.

The conjecture holds for trees and graphs of maximal degree at most 2 (Exercises!)

Theorem 16.

(Galvin 1995)
Bipartite graphs are $\Delta(G)$-edge-choosable.