(Chapter 5) Coloring: definitions and elementary concepts

Definition 1.

A **(feasible)** k-**coloring of vertices** *of a given graph* G *is a mapping* $c\colon V(G) \to \{1, 2, \ldots, k\}$ *for which* $\{u, v\} \in E(G)$ *implies* $c(u) \neq c(v)$, *for any two vertices* $u, v \in V(G)$ *and some natural number* $k \in \mathbb{N}$. *We say "the vertex coloring* c *uses at most* k *colors" and* G *is* k*-colorable.* *The* **chromatic number of a graph** G, *denoted by* $\chi(G)$, *is the smallest natural number* k *such that* G *is* k*-colorable.*

Definition 2.

A **(feasible)** k–**coloring of edges** *of a given graph* G *is a mapping* $c\colon E(G) \to \{1, 2, \ldots, k\}$ *for which* $e_1 \cap e_2 \neq \emptyset$ *implies* $c(e_1) \neq c(e_2)$, *for any two edges* $e_1, 2_2 \in E(G)$ *and some natural number* $k \in \mathbb{N}$.*We say "the edge coloring* c *uses at most* k *colors" and* G *is* k*-edge colorable.* *The* **chromatic index of a graph** G, *denoted by* $\chi'(G)$, *is the smallest natural number* k *such that* G *is* k*-edge-colorable.*

(5) Coloring: examples and simple bounds

Definition 3.

The **line graph** $L(G)$ *of a given graph* G *is a graph with vertex set* $E(G)$, *i.e.* $V(L(G)) = E(G)$, *such that two vertices of* $L(G)$ *are connected by an edge iff the correspondig edges in* G *have a common endvertex.*

Observations:

- $\chi'(G) = \chi(L(G))$ holds for any grpah G
- $\chi(K_n) = n$, for any $n \in \mathbb{N}$
- For any graph G the inequality $\chi(G) \geq \omega(G)$ holds, where $\omega(G)$ is the clique number of G, i.e. the cardinality of the largest clique in G.
- $\chi(G) = 2$ holds iff G is a bipartite graph.
- For any graph G the inequality $\chi(G) \geq \frac{|G|}{\alpha(G)}$ holds, where $\alpha(G)$ is the stability number of G, i.e. the cardinality of the largest stable set in G.

(5) Coloring: elementary results

Definition 4.

The **clique partition number** $\theta(G)$ *of a given graph* G *is the smallest number of cliques in which* $V(G)$ *can be partitioned.*

Observation: $\chi(G) = \theta(G^c)$ holds for any graph G.

Theorem 1.

(The four color theorem)
Every planar graph is 4 *colorable, i.e. for every planar graph there exists a feasible* 4*-coloring of its vertices.*

K. Appel and W. Haken, Every Planar Map is Four-Colorable, *Contemporary Mathematics* **98**, 1989
N. Robertson, D. Sanders, P.D. Seymour and R. Thomas, The four-color theorem, *Journal of Combinatorial Theory (B)* **70 (1)**, 1997, 2–44.

Theorem 2.

(Grötsch 1959)
Every triangle-free planar graph is 3 *colorable, i.e.* $\chi(G) \leq 3$ *holds for every such graph* G*.*

(5) Coloring: the graph k-colorability problem ($GC(k)$)

Instance: A graph $G = (V.E)$, a natural number $k \in \mathbb{N}$.
Question: Is G k-colorable?

Computational complexity results:
$GC(2)$ is polynomially solvable.
For any $k \in \mathbb{N}$, $GC(k)$ is polynomially solvable if $\Delta(G) \leq 3$ (Brooks, 1941)
$GC(k)$ is NP-hard for every fixed k, $k \geq 3$.
$GC(3)$ is NP-hard also for planar graphs G with $\Delta(G) \leq 4$.

M.R. Garey and D.S. Johnson, *Computers and intractability: a guide to the theory of NP-completeness*, W.H. Freeman and Company, NY, 1979.

Proposition 3.

(An upper bound)

Every graph G with m edges satisfies $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$.

(5) Coloring: the greedy algorithm for vertex coloring (GREEDY)

Input: A graph $G = (V, E)$
Output: A feasible k-coloring $c\colon V \to \mathbb{N}$ for some (unknown) $k \in \mathbb{N}$

Step 1 Enumerate vertices of G arbitrarily as $v_1, \ldots, v_n$, where $n := |V|$.

Step 2 For $i = 1$ to n do:

color v_i with the smallest color $c(v_i) \in \mathbb{N}$ such that $c(v_j) \neq c(v_i)$ for all $v_j \in N(v_i) \cap \{v_1, \ldots v_{i-1}\}$.

Remarks:

(1) GREEDY never uses more that $\Delta(G) + 1$ colors.

(2) If (a) $G = K_n$ or (b) $G = C_n$ and n is odd, then GREEDY produces a $\chi(G)$-coloring.

(3) In general $\Delta(G) + 1$ is a loose bound for $\chi(G)$ and also for the number of colors needed by the greedy algorithm.

(5) Coloring: the greedy algorithm for vertex coloring (GREEDY)

GREEDY colors v_i by $c(v_i) = deg_{G[\{v_1, v_2, \ldots, v_i\}]}(v_i) + 1$.

Idea: Choose an approriate ordering of vertices in Step 1 such that the number of colors used by GREEDY is small!

Example of an ordering: Choose v_n such that $\delta(G) = deg(v_n)$ first, then choose v_{n-1} as the vertex with the smallest degree in $G[V \setminus \{v_n\}]$, and so on.

Definition 5.

The **coloring number of** G*, denoted by* $col(G)$*, is the smallest number* $k \in \mathbb{N}$ *such that there exists an ordering of vertices in* G *with the property that every vertex is preceeded by fewer than* k *of its neighbors.*

Proposition 4.

Every graph satisfies $\chi(G) \leq col(G) = \max\{\delta(H)\colon H \subseteq G\} + 1$.

Corollary 5.

Every graph has a subgraph of minimum degree at least $\chi(G) - 1$.

(5) Coloring: the theorem of Brooks

Theorem 6.

(Brooks 1941)
Let G be a connected graph. If G is neither complete nor an odd cycle than $\chi(G) \leq \Delta(G)$.

Questions: Is there any relationship between $\chi(G)$, $\kappa(G)$ and $\delta(G)$?

No, neither implies a large $\chi(G)$ a large $\kappa(G)$ or $\delta(G)$, nor imply large values $\kappa(G)$ or $\delta(G)$ a large $\chi(G)$.

Theorem 7.

(Erdös 1959)
For all $k \in \mathbb{N}$, *there exists a graph G with girth* $g(G) > k$ *and chromatic number* $\chi(G) > k$.

Erdös' theorem shows that a large chromatic number can occur as a purely global phenomenon in a graph!