Coloring 1

Question: 1) Assume $\chi(G)=1$.
What kind of graph is G?

2) Assume $\chi'(G)=1$

Observation: $\chi(G) \geq \dfrac{|G|}{\alpha(G)}$

Why? Let c be a coloring of G with $\chi(G)$ colors from $\{1,2,\dots,\chi(G)\}$ and let $S_i := c^{-1}(\{i\}),\ \forall i \in \overline{1,\chi(G)}$.

Then $V(G) = \bigcup\limits_{i=1}^{\chi(G)} S_i \Rightarrow |G| = |V(G)| = \sum\limits_{i=1}^{\chi(G)} |S_i| \leq \chi(G)\alpha(G)$

$\Rightarrow \chi(G) \geq \dfrac{|G|}{\alpha(G)}$.

Question: How good is the bound above?

In general this bound can be arbitrarily bad, just as the clique number bound $\rightarrow$ exercises

There are particular cases in which the bound is tight, e.g.

K_n or $K_{n,n}$

Def Let c be a feasible k-coloring of G, $c: V \to \{1, 2, \dots, k\}$. A color class is a maximal set of vertices colored by the same color (maximal with respect to set inclusion), i.e. the color classes are $c^{-1}(\{i\})$, $\forall\, i \in \overline{1,k}$.

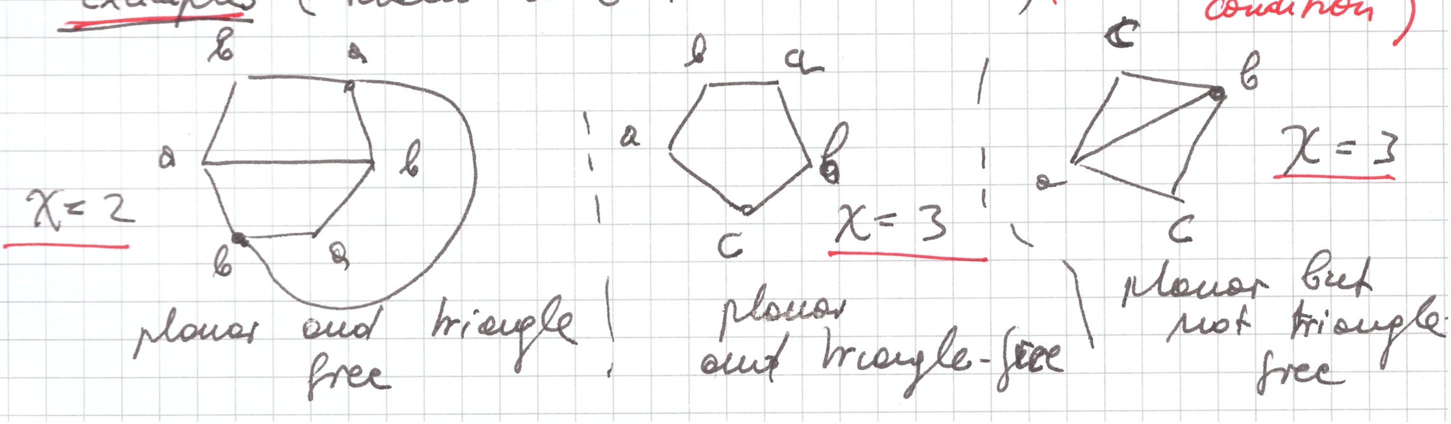
Clearly $c^{-1}(\{i\})$ is a stable set in G and $c^{-1}(\{1\}), c^{-1}(\{2\}), \dots, c^{-1}(\{k\})$ is a partition of $V(G)$ with cardinality k. In particular ~~we can find~~ there exist such a partition of $V(G)$ in stable sets with cardinality $\chi(G)$.

Thus we obtain an equivalent definition of $\chi(G)$:

Def $\chi(G)$ is the smallest number of stable sets in G in which $V(G)$ can be partitioned, or equivalently $\chi(G)$ is the smallest number of cliques in G^c in which $V(G)$ can be partitioned.

The latter number is called clique partition number of G^c, denoted by $\theta(G^c)$. Thus $\chi(G) = \theta(G^c)$.

Examples (related to Grötzsch's theorem) (only a sufficient condition)



Proposition 3 (An upper bound)

$\forall$ graph G: $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$ where $m := |E(G)|$

Proof Let c be a coloring with $\chi(G) =: k$ colors. Then G has at least between any two color classes, otherwise we could use the same color for both color classes. Since there are $\frac{k(k-1)}{2}$ pairs of color classes we get $m \geq \frac{k(k-1)}{2} \Rightarrow 2m \geq (k-1)k$.

Solve this inequality for k: $k^2 - k - 2m \leq 0$

The roots of $k^2 - k - 2m = 0$ are $k_{1,2} = \frac{1 \pm \sqrt{1+8m}}{2}$

and only for $k \in (k_1, k_2)$ we get $k^2 - k - 2m < 0$

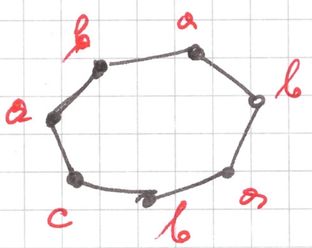
Thus $k^2 - k - 2m \leq 0 \Rightarrow k \leq k_2 = \frac{1 + \sqrt{1+8m}}{2} = \frac{1}{2} + \sqrt{2m+}$

Remarks on the greedy algorithm for vertex coloring

(1) GREEDY never uses more than $\Delta(G)+1$ colors (trivial: the vertex to be colored cannot have more than $\Delta(G)$ neighbors among the already colored vertices)

(2) GREEDY colors K_n with n colors (trivial)

GREEDY on C_{2k+1}



(3) $\Delta(G)+1$ is a loose bound for ~~???~~ $\chi(G)$ and also for the nr. of colors used by GREEDY $\rightarrow$ exercises

Equivalent definition of the coloring number:

The smallest possible number of colors, $col(G)$, used by ~~the~~ GREEDY applied for any ordering of the vertices of the graph.

Proposition 4, $\chi(G) \leq col(G) = \max\{\delta(H) : H \subseteq G\} + 1$

Proof: $\chi(G) \leq col(G)$ is trivial because there exists a feasible coloring with $col(G)$ colors

So we show $col(G) = \max\{\delta(H) : H \subseteq G\} + 1$.

"$\leq$" The enumeration / order of vertices mentioned above (in the equiv. def of $col(G)$) show that $col(G) \leq \max\{\delta(H) : H \subseteq G\} + 1$ (homework)

"$\geq$" $col(G) \geq col(H)$ holds $\forall H \subseteq G$
(by applying GREEDY in H with the same order of vertices as in G, the number of ~~used~~ colors used in H can only decrease as compared to the nr. of used colors in G)

Moreover $col(H) \geq \delta(H) + 1$ since the "back-degree" of the last vertex is just its ordinary degree and this is at least $\delta(H)$ so the last vertex will get the color $\delta(H) + 1$.

so $col(G) \geq col(H) \geq \delta(H) + 1 \quad \forall H \subseteq G$

$\Rightarrow col(G) \geq \max\{\delta(H) : H \subseteq G\} + 1$ $\boxtimes$