(Chapter 6) Chordal graphs: definitions and characterisations

Definition 7.

(Hajnal, Surányi 1958)
A **graph** *G is called* **chordal** *iff G contains no cycle of length k, $k \geq 4$, as an induced subgraph. Equivalently, G is chordal iff there is a* **chord** *for every cycle of length k, $k \geq 4$, in G, i.e. there is an edge $\{x, y\} \in E(G)$ connecting two non-consecutive vertices in C, $x, y \in V(C)$, $\{x, y\} \notin E(C)$.*

Trivial example: complete graph K_n, $n \in \mathbb{N}$, are chordal.

Observation: A bipartite graph is chordal iff it is a forest.

Lemma 10.

(Dirac 1961)
A graph G is chordal iff every separator which is minimal with respect to set-inclusion induces a clique in G.

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Definition 8.

Let G be a graph and let x be a vertex in $V(G)$. x is called a **simplicial vertex** *if the neighbors $N_G(v)$ induce a clique in G.*

Lemma 11.

(Dirac 1961)
Let G be a chordal graph G. Then G contains a simplicial vertex. If G is not a complete graph, then it contains two non-adjacent simplicial vertices.

Theorem 12.

(Berge 1961, Hajnal, Surányi 1958)
Chordal graphs and their complements are perfect.

Definition 9.

A labelling (or a total order) $\sigma\colon V(G) \to \{1, 2, \ldots, |G|\}$ on the vertex set $V(G)$ of a graph G is called a **perfect vertex elimination scheme (PSE)** *iff every $v \in V(G)$ is a simplicial vertex in $G[\{u \in V(G)\colon \sigma(u) \geq \sigma(v)\}]$, i.e. iff the so-called* **upper neighborhood** *$UN(v)$ of v, builds a clique in G, for every $v \in V(G)$, where $UN(v) = \{u \in N(v)\colon \sigma(u) \geq \sigma(v)\}$.*

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Theorem 13.

(Fulkerson, Gross 1965)
A graph G is chordal iff it has a perfect vertex elimination scheme.

Definition 10.

A **maximum adjacency order (MAO)** *in a graph G is a labelling $\sigma\colon V(G) \to \{1, 2, \ldots, |G|\}$ constructed iteratively as follows. Set $\sigma(v_1) = 1$ for some arbitrarily chosen vertex v_1 and repeat iteratively the following step: for all $i = 1$ to $n-1$ set $\sigma(v) = i+1$ for some $v \in V(G) \setminus \{v_1, \ldots, v_i\}$ with the maximum number of neighbors among in $\{v_1, \ldots, v_i\}$, i.e.*
$deg_{\{v_1,\ldots,v_i\}}(v) = \max\left\{deg_{\{v_1,\ldots,v_i\}}(x)\colon x \in V(G) \setminus \{v_1, \ldots, v_i\}\right\}$.

Remark: MAO is not uniquely defined.

Proposition 14.

A labelling $\sigma\colon V(G) \to \{1, 2, \ldots, |G|\}$ is a PES in a graph G iff the following implication holds: for any $v_i, v_j \in V(G)$, $v_i \neq v_j$, the existence of a v_i-v_j-path P such that all $\sigma(u) < \min\{\sigma(v_i), \sigma(v_j)\}$ holds for all its internal nodes u ($u \in \mathring{P}$), implies $\{v_i, v_j\} \in E(G)$.

(6) Chordal graphs: characterisation and recognition

Proposition 15.

Let G be a chordal graph and $\sigma\colon V(G) \to \{1, 2, \ldots, n\}$, $n := |G|$, be a MAO in G with $\sigma(v_i) = i$, for all $i \in \{1, 2, \ldots, n\}$.
Then the labelling $\tilde{\sigma}\colon V(G) \to \{1, 2, \ldots, |G|\}$ with $\tilde{\sigma}(v_i) = n - i + 1$, for all $i \in \{1, 2, \ldots, n\}$, is a PES in G.

The recognition problem for chordal graphs
Input: A graph $G = (V, E)$.
Question: Is G chordal?

Theorem 16.

The recognition problem for a chordal graph G can be solved in $O(m + n \log n)$ time, where $n := |V(G)|$ and $m := |E(G)|$.

(6) Chordal graphs: characterisation and recognition

Proposition 17.

Let $G = (V, E)$ be a graph and let $\sigma\colon V(G) \to \{1, 2, \ldots, n\}$, $n := |G|$, be a MAO in G with $\sigma(v_i) = i$, for all $i \in \{1, 2, \ldots, n\}$. Consider the labelling $\tilde{\sigma}\colon V(G) \to \{1, 2, \ldots, |G|\}$ with $\tilde{\sigma}(v_i) = n - i + 1$, for all $i \in \{1, 2, \ldots, n\}$. The following equivalence holds: G is chordal iff $\tilde{\sigma}$ is a PES in G.

A generic algorithm to decide whether a graph G is chordal
Input: $G = (V, E)$ with $n = |V|$, $m = |E|$, $\forall v \in V$, $Adj(v) \subset V$ the set of all vertices adjacent to v.
Output: TRUE, if G is chordal, and FALSE, otherwise.

(1) Determine an MAO $\sigma\colon V \to \overline{1, n}$ in G.
(Can be done in $O(m + nlogn)$ time by using Fibonacci heaps.)

(2) Compute $\tilde{\sigma}\colon V \to \overline{1, n}$ with $\tilde{\sigma}(\sigma^{-1}(n - i + 1)) = i$ for $i \in \overline{1, n}$.)
(Can be done in $O(m)$ time.)

(3) Check whether $\tilde{\sigma}$ is a PES in G.
(Can be done in $O(n + m)$ time by using the leader f_v of the upper neighborhood $UN(v)$, where $f_v := argmin\{\tilde{\sigma}(u)\colon u \in UN(v)\}$ for all $v \in V$.)

(6) Chordal graphs: computation of graph invariants and optimization problems

Proposition 18.

Let G be a chordal graph with a PES σ. All maximal cliques of G can be found among $(C_v)_{v\in V(G)}$, where $C_v := \{v\} \cup UN(v)$, for all $v \in V(G)$. Thus G has at most n maximal cliques. Moreover G has exactly n maximal cliques iff $E(G) = \emptyset$.

The largest clique in G is C_{v^*} with $|C_v^*| = \max\{|C_v|: v \in V(G)\}$ and $\omega(G) = |C_v^*|$.

Proposition 19.

Let G be a chordal graph with a PES σ. Construct the set of vertices $S := \{s_1, \ldots, s_l\}$ iteratively as follows for some $l \in \mathbb{N}$: $s_1 = \sigma^{-1}(1)$ and $\forall k \leq 2$, $k \in \mathbb{N}$ set

$$s_k := argmin\left\{\sigma(v)\colon \sigma(v) > \sigma(s_{k-1}) \text{ and } v \notin \cup_{i=1}^{k-1} UN(s_i)\right\}$$

als long as the set on the right hand side of the above equality is nonempty.
Then S is a **maximum stable set** *in G, $(C_{s_k})_{k\in\overline{1,l}}$ is* **a clique cover of minimum cardinality** *in G, and $\alpha(G) = \theta(G) = l$ holds.*

(6) Comparability graphs

Definition 11.

A graph G is called a **comparability graph** *if there exists an order relation $P_{\leq}$ on $V(G)$ such that $\forall \{x, y\} \in E(G)$ the vertices x and y are comparable in $P_{\leq}$, i.e. $xP_{\leq}y$ or $yP_{\leq}x$.*

Lemma 20.

G is a comparability graph iff G is **transitively orientable**, *i.e. there exists an orientation $o\colon E(G) \to V(G) \times V(G)$ which assigns an orientation to every edge $\{u, v\}$ (making a directed edge, or an arc, (u, v) or (v, u) out of $\{u, v\}$) such that $(x, y) \in O$ and $(x, z) \in O$ implies $(x, z) \in O$.*

Example: Any bipartite graph $G = (U \dot{\cup} V, E)$ is a comparability graphs. A partial order $P_{\leq}$ on $U \dot{\cup} V$ can be defined by

$$uP_{\leq}v \Leftrightarrow (\{u, v\} \in E, u \in U, y \in V) \text{ or } (u = v)\,.$$

An orientation O is in Lemma 20 is obtained by orienting every edge from U to V.

(6) Comparability graphs: continued

Recall Dilworth's theorem and its dual:

Theorem 21.

(Dilworths's theorem)
Let $P_{\leq}$ be a partial order on a finite set M. The length of a longest antichain in $(M, O_{\leq})$ equals the minimum number of chains in a chain decomposition of M.

Theorem 22.

Let $P_{\leq}$ be a partial order on a finite set M. The length of a longest chain in $(M, O_{\leq})$ equals the minimum number of antichains needed to cover M by antichains.

Corollary 23.

Comparability graphs are perfect graphs.

Observation: Let G be a comparability graph and let $P_{\leq}$ be a partial order as in Definition 11. Then cliques in G correspond to chains in $(V(G), P_{\leq})$ and stable sets in G correspond to antichains in $(V(G), P_{\leq})$.

(6) Comparability graphs: continued

Proposition 24.

Comparability graphs with n vertices and m edges can be recognized in $O(n+m)$ time.

R.M. McConnell and J. Spinrad, Modular decomposition and transitive orientation, *Discrete Mathematics* **201**, 1999, 189-241.

(6) Interval graphs

Definition 12.

G is called an interval graph iff G is the intersection graph of a family of closed intervals $([a_i, b_i])_{i \in I}$ in $\mathbb{R}$, where $a_i \leq b_i$, $\forall i \in I$. In other words, G is an interval graph, iff there exists a bijection $f : V(G) \to I$ such that $\forall x, y \in V(G)$, $x \neq y$, the equivalence $\{x, y\} \in E(G) \Leftrightarrow f(x) \cap f(y) \neq \emptyset$ holds.

Theorem 25.

(Gilmore, Hoffman 1964)
For a given graph G the following statements are equivalent:

(i) *G is an interval graph,*

(ii) *G does not contain an induced cycle of length at least 4 and the complement $\bar{G}$ is a comparability graph,*

(iii *the maximum cliques of G can be ordered linearly such that $\forall v \in V(G)$ the maximum cliques containing v build a contiguous interval in that order. In other words, there exists a sequence C_1, C_2, ..., C_k of all maximum cliques in G, for some $k \in \mathbb{N}$, such that $\forall v \in V(G)$, there exist $i_v \in \overline{1, k}$ and an $n_v \in \mathbb{N}$ with $i_v + n_v \leq k$ such that the maximum cliques of G containing V are C_{i_v},C_{i_v+1},*

(6) Interval graphs: continued

Theorem 26.

Interval graphs are perfect graphs.

Proposition 27.

Interval graphs with n vertices and m edges can be recognized in $O(n+m)$ time.

K. Simon, A simple linear time algorithm to recognize interval graphs, H. Bieri and H. Noltemeier, eds., Springer, *Lecture Notes in Computer Science* **553**, 1991, 289-308.