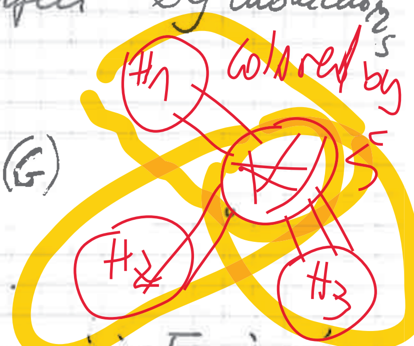
a clique in G.

Let H_i, $i \in I$, $|I| \geq 2$, be the connected components of $G - S$.

Then the chordal graphs $G[H_i \cup S]$ are perfect by induction assumption $\forall i \in I$.



Observe now that this implies $\chi(G) = \omega(G)$

Indeed, consider a clique C with $|C| = \omega(G)$.

Since there is no edge connecting H_i, H_j for $i, j \in I$, $i \neq j$.

$\exists i \in I$ such that $C \subseteq H_i \cup S$, $\Rightarrow$ $\omega(G) = \omega(G[H_i \cup S])$

$\underset{\text{ind. assumption}}{=} \chi(G[H_i \cup S])$. Color now $G[H_i \cup S]$ by $\chi(G[H_i \cup S]) = \omega(G)$ colors. Denote this coloring by c.

$\forall j \in I$, $j \neq i$ extend the coloring of $G[H_i \cup S]$ to a coloring of $G[H_j \cup S]$ using $\chi(G[H_j \cup S]) = \omega(G[H_j \cup S])$

$\leq \omega(G) = \chi(G[H_i \cup S]) =: \chi$ colors as follows: use the same colors as in the coloring of $G[H_i \cup S]$ for the vertices in S and the remaining $\{1, 2, \dots \chi\} \setminus \{c(x) \mid x \in S\}$.

This can be done because $\chi(G[H_j \cup S]) \leq \chi$.

Since there are no edges joining H_i, H_j for $i \neq j$, this is a feasible coloring for the whole graph G which uses χ colors

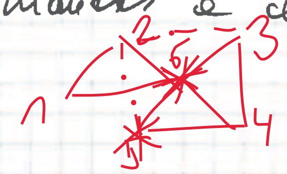
$\Rightarrow$ $\chi(G) \leq \chi = \omega(G)$ holds, and this completes the proof of the perfectness of a chordal graph.

The perfectness of the complement of a chordal graph follows immediately from the perfect graph theorem. ∎

The recognition of chordal graphs

We start with an algorithmic characterisation of chordal graphs.

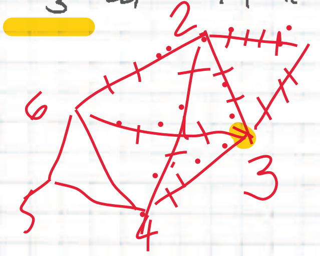
Def Let $G=(V,E)$ be a graph with $n:=|G|$.
A ~~(bijective)~~ labelling $\sigma: V(G) \to \{1,2,\dots,n\}$ is called perfect vertex elimination scheme (PES) iff $\forall v \in V(G)$ v is a simplicial vertex in $G[\{u \in V : \sigma(u) \geq \sigma(v)\}]$, i.e. $\forall v \in V$ the upper neighborhood of v with respect to σ, denoted by $UN_\sigma(v)$, induces a clique in G, where



$$UN_\sigma(v) = \{u \in N(v) : \sigma(u) > \sigma(v)\}.$$

A labelling is by definition a bijection.

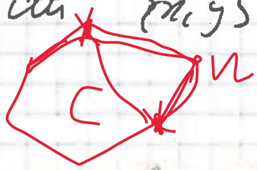
Observation: If G is chordal, then there exists a PES in G.
(Find a simplicial vertex v_1, consider $G \setminus v_1$ which is chordal ~~perfect~~, find a simplicial vertex v_2 in $G \setminus v_1$, consider $G \setminus \{v_1, v_2\}$ which is chordal, find a simplicial vertex v_3 in $G \setminus \{v_1, v_2\}$ and so on...



The converse also holds:

Proposition (Fulkerson, Gross 1965)
A graph G is chordal iff there is a PES in G.

Proof $\Rightarrow$) holds due to the above observation.
We show $\Leftarrow$). Assume $\exists$ PES $\sigma: V(G) \to \{1,2,\dots \underbrace{|G|}_{=:n}\}$ in G and show that G is chordal. Consider a cycle C in G.
Let $u \in V(C)$ be such that $\sigma(u) = \min\{\sigma(v) : v \in V(C)\}$
Then $V(C) \cap UN_\sigma(u)$ induces a clique in $G \Rightarrow$
$\exists$ two vertices $x, y \in V(C)$ with $\{x,y\} \in E(G) \setminus E(C)$ as soon as $|V(C)| \geq 4$.



This proposition suggests a simple algorithm to solve the recognition problem for chordal graphs:

Recognition Problem for Chordal Graphs

Input : $G = (V, E)$ a graph
Question : is G chordal ?

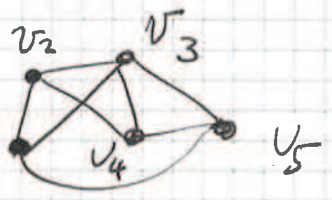
Try to find a sequence of vertices $v_1, \dots v_n$ in G where $n := |G|$ as in the proof of the previous observation. If such a sequence is found then it represents a PES (this ordering of vertices represent a PES). Otherwise, if in some iteration $i_{>1}$ no simplicial vertex in $G \setminus \{v_1 \dots v_{i-1}\}$ can be found (if $i=0$ we have $G \setminus \{v_1 \dots v_{i-1}\}$ would simply be G) then conclude that G is not chordal.

If the graph its save by the adjacency lists of its vertices the above approach takes $O(|V(G)|^2 |E(G)|)$ time. (homework or exercise)

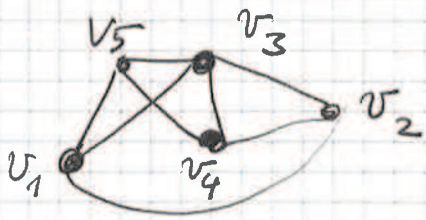
But we can do better than that by using the so-called maximum adjacency order.

<u>Def</u> ~~The~~ max adjacency order (MAD) is a labelling of vertices $\sigma : V(G) \to \{1, 2, \dots n = |G|\}$ in a graph G is defined by starting with an arbitrary vertex $\sigma^{-1}(1) = v_1$ and repeating iteratively the following step:
for all $i = 1$ to n set $\sigma^{-1}(i+1) = v$ for some v with max number of neighbors among $\{v_1, \dots, v_i\}$.

<u>Example</u>



or



with $v_i = \sigma^{-1}(i)$ $\forall i \in \overline{1,5}$

<u>Remark</u>: MAD is not uniquely defined

Next we show that in a chordal graph with MAD σ the reverse labelling $\tilde{\sigma} : V(G) \to \{1, 2, \dots, n\}$, $v_{n-i+1} \mapsto i$
i.e. $v_n, v_{n-1}, \dots, v_1$ is a PES. $\tilde{\sigma}(v_{n-i+1}) = i$

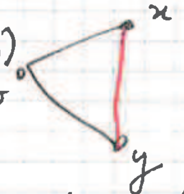
Here we denote : $v_i := \sigma^{-1}(i)$, $\forall i \in \overline{1,n}$ is a PES.

Prior to proving that we show the following proposition.

Proposition A labelling $\sigma: V(G) \to \{1,2,\ldots,n\}$ is a PES in G iff $\forall\, v_i, v_j \in V(G)$, the existence of a v_i–v_j–path P with $\sigma(u) < \min\{\sigma(v_i), \sigma(v_j)\}$ $\forall\, u \in P^\circ$ (i.e. for every inner vertex of P) implies $\{v_i, v_j\} \in E(G)$.

Proof $\Leftarrow$) Assume "$\exists$ v_i-v_j-path P with ~~$\sigma(u) < \min\{\sigma(v_i)$~~ $\sigma(u) < \min\{\sigma(v_i), \sigma(v_j)\}$ $\forall\, u \in \mathring{P} \Longrightarrow \{v_i, v_j\} \in E(G)$"

Show that σ is a PSE, i.e. $UN_\sigma(v)$ induces a clique in G, $\forall\, v \in V(G)$. Indeed consider a $v \in V(G)$ and the upper neighborhood $UN_\sigma(v)$. Let $x, y \in UN_\sigma(v)$.

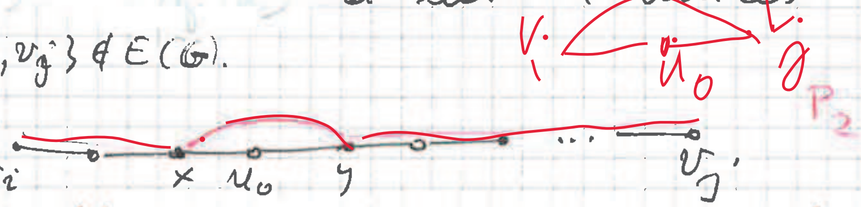
We have $P = (x, v, y)$  and $\sigma(v) < \sigma(x)$, $\sigma(v) < \sigma(y)$

The assumption implies $\{x, y\} \in E(G)$

$\Longrightarrow$) Assume σ is a PSE.

Assume by contradiction that a v_i–v_j–path P with $\sigma(u) < \min\{\sigma(v_j), \sigma(v_i)\}$, $\forall\, u \in \mathring{P}$, exists and $\{v_i, v_j\} \notin E(G)$ for some $v_i, v_j \in V(G)$, $v_i \neq v_j$. Consider a shortest path P_1 with these properties.

Let $u_0 := \operatorname{argmin}\{\sigma(v) : v \in P_1\}$. Then $u_0 \in \mathring{P}_1$ and P_1 has at least 4 vertices (incl. v_i, v_j) because $\{v_i, v_j\} \notin E(G)$.

P_1 

But the neighbors x, y of u_0 in P_1 belong to $UN_\sigma(u_0)$ and are therefore connected by an edge $\{x, y\} \in E(G)$. But then the path $P_2 = (v_i, \ldots, x, y, \ldots, v_j)$ is shorter than P_1 and has the properties listed above, a contradiction ↯

$\boxed{\text{M}}$

Proposition ~~Let~~ G ~~be~~ a chordal graph and $\sigma: V(G)\to\{1,2,\dots,n\}$ be an MAO in G with $\sigma(v_i)=i$, $\forall i\in\overline{1,n}$, $n:=|V(G)|$.

Then $v_n, v_{n-1},\dots,v_1$ is a PES in G.

Proof Let $\tilde\sigma: V(G)\to\{1,2,\dots,n\}$ with $\tilde\sigma(v_i)=n-i+1$, $\forall i\in\overline{1,n}$

We observe that $\tilde\sigma$ has the following property:

$$(*)\quad \forall u,v,w\in V(G) \quad \left.\begin{array}{l}\tilde\sigma(u)<\tilde\sigma(v)<\tilde\sigma(w)\\ \{u,w\}\in E(G),\ \{v,w\}\notin E(G)\end{array}\right\} \Longrightarrow \begin{array}{l}\exists z\in V(G)\\ \text{with } \tilde\sigma(v)<\tilde\sigma(z)\\ \{v,z\}\in E(G)\\ \text{and } \{u,z\}\notin E(G)\end{array}$$

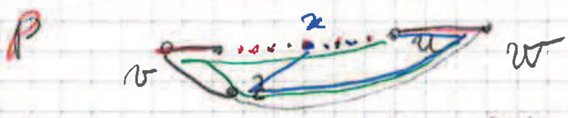
Indeed, the MAO label of v is smaller than that of u ($\sigma(v)<\sigma(u)$) although u is adjacent to the already labelled w ($\sigma(w) > \sigma(u)$ and $\sigma(w) > \sigma(v)$) and v is not. Since v has at least as many neighbors with MAO label smaller than $\sigma(v)$ as the number of neighbors of u with PSE label $<\sigma(v)$. Thus there must exist a neighbor z of v with $\sigma(z)<\sigma(v)$ which is not a neighbor of u.

end proof of (*)

Assume now that $\tilde\sigma$ is not a PSE. According to the previous proposition $\exists$ 2 vertices $v,w\in V(G)$, $\{v,w\}\notin E(G)$ which are connected by a v-w-path P with $\tilde\sigma(x)<\min\{\tilde\sigma(v),\tilde\sigma(w)\}$ $\forall x\in\mathring{P}$. Choose two such vertices v, w such that $\tilde\sigma(v)$ is as large as possible, and if there are many possibilities for w, choose it also such as $\tilde\sigma(w)$ is as large as possible. Let P be the shortest v-w-path ~~with the~~ above property ($\forall x\in\mathring{P}$ $\tilde\sigma(x)<\min\{\tilde\sigma(v),\tilde\sigma(w)\}$).

Since $\{v,w\}\notin E(G)$, P contains at least one inner vertex.

Let $u\in\mathring{P}$ with $\{u,w\}\in E(G)$



We have $\tilde\sigma(v)>\tilde\sigma(w)>\tilde\sigma(u)$

and $\{u,w\}\in E(G)$ $\{v,w\}\notin E(G)$

- According to (*) $\exists\, z\in V(G)$ with $\tilde\sigma(v)<\tilde\sigma(z)$, $\{v,z\}\in E(G)$, $\{u,z\}\notin E(G)$